

# Boundedness of Commutators on Hardy Spaces over Metric Measure Spaces of Non-homogeneous Type

Haibo Lin, Suqing Wu and Dachun Yang\*

**Abstract.** Let  $(\mathcal{X}, d, \mu)$  be a metric measure space satisfying the so-called upper doubling condition and the geometrically doubling condition. Let  $T$  be a Calderón-Zygmund operator with kernel satisfying only the size condition and some Hörmander-type condition, and  $b \in \widetilde{\text{RBMO}}(\mu)$  (the regularized BMO space with the discrete coefficient). In this paper, the authors establish the boundedness of the commutator  $T_b := bT - Tb$  generated by  $T$  and  $b$  from the atomic Hardy space  $\widetilde{H}^1(\mu)$  with the discrete coefficient into the weak Lebesgue space  $L^{1,\infty}(\mu)$ . The boundedness of the commutator generated by the generalized fractional integral  $T_\alpha$  ( $\alpha \in (0, 1)$ ) and the  $\widetilde{\text{RBMO}}(\mu)$  function from  $\widetilde{H}^1(\mu)$  into  $L^{1/(1-\alpha),\infty}(\mu)$  is also presented. Moreover, by an interpolation theorem for sublinear operators, the authors show that the commutator  $T_b$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ .

## 1 Introduction

The classical theory of Calderón-Zygmund operators originated from the study of the convolution operator with singular kernel on  $\mathbb{R}$ . From then on, it has become one of the core research areas in harmonic analysis and has been developed into a large branch of analysis on metric spaces, among which, one of the most useful underlying spaces is the space of homogeneous type introduced by Coifman and Weiss [7, 8]. Recall that a quasi-metric space  $(\mathcal{X}, d)$  equipped with a non-negative measure  $\mu$  is called a *space of homogeneous type* in the sense of Coifman and Weiss [7, 8] if  $(\mathcal{X}, d, \mu)$  satisfies the *measure doubling condition*: there exists a positive constant  $C_{(\mu)}$  such that, for all balls  $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$  with  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ ,

$$(1.1) \quad \mu(B(x, 2r)) \leq C_{(\mu)} \mu(B(x, r)).$$

---

2010 *Mathematics Subject Classification.* Primary 47B47; Secondary 42B20, 42B35, 30L99.

*Key words and phrases.* geometrically doubling measure, upper doubling measure, metric measure space of non-homogeneous type, commutator, Calderón-Zygmund operator, fractional integral,  $\widetilde{\text{RBMO}}$  function, Hardy space.

Haibo Lin is supported by the National Natural Science Foundation of China (Grant Nos. 11301534 and 11471042) and Da Bei Nong Education Fund (Grant No. 1101-2413002). Dachun Yang is supported by the National Natural Science Foundation of China (Grant Nos. 11571039 and 11361020), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2013YB60 and 2014KJJCA10).

\*Corresponding author

As was well known, the space of homogeneous type is a natural setting for Calderón-Zygmund operators and function spaces. Euclidean spaces equipped with Lebesgue measures, Euclidean spaces equipped with weighted Radon measures satisfying the doubling condition (1.1), Heisenberg groups equipped with left-variant Haar measures are all the typical examples of spaces of homogeneous type.

On the other hand, in the last two decades, many classical results concerning the Calderón-Zygmund operators and function spaces have been proved still valid for metric spaces equipped with *non-doubling measures*; see, for example, [29, 30, 38, 39, 40, 41, 42, 5, 6, 15, 18]. In particular, let  $\mu$  be a non-negative Radon measure on  $\mathbb{R}^d$  which only satisfies the *polynomial growth condition* that there exist some positive constant  $C_0$  and  $n \in (0, d]$  such that, for all  $x \in \mathbb{R}^d$  and  $r \in (0, \infty)$ ,

$$(1.2) \quad \mu(B(x, r)) \leq C_0 r^n,$$

where  $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$ . Such a measure does not need to satisfy the doubling condition (1.1). Tolsa [38, 41] introduced the atomic Hardy space  $H_{\text{atb}}^{1,q}(\mu)$ , for  $q \in (1, \infty]$ , and its dual space,  $\text{RBMO}(\mu)$ , the *space of functions with regularized bounded mean oscillation*, with respect to  $\mu$  as in (1.2), and proved that Calderón-Zygmund operators are bounded from  $H_{\text{atb}}^{1,q}(\mu)$  into  $L^1(\mu)$ . In [15], Hu et al. established an equivalent characterization of  $H_{\text{atb}}^{1,q}(\mu)$  to obtain the boundedness on  $L^p(\mu)$  of commutators and their endpoint estimates. More research on function spaces, mainly on Morrey spaces, and their applications related to non-doubling measures can be found in, for example, [13, 32, 34, 35]. We point out that the analysis on such non-doubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [40, 42].

However, as was pointed out by Hytönen in [19], the measure  $\mu$  satisfying the polynomial growth condition is different from, not general than, the doubling measure. Hytönen [19] introduced a new class of metric measure spaces satisfying both the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 2.1 and 2.4 below), which are also simply called *metric measure spaces of non-homogeneous type*. These metric measure spaces of non-homogeneous type include both metric measure spaces of homogeneous type and metric measure spaces equipped with non-doubling measures as special cases. We mention that several equivalent characterizations for the upper doubling condition were recently established by Tan and Li [36, 37].

From now on, we always assume that  $(\mathcal{X}, d, \mu)$  is a metric measure space of non-homogeneous type in the sense of Hytönen [19]. In this new setting, Hytönen [19] introduced the space  $\text{RBMO}(\mu)$  and established the corresponding John-Nirenberg inequality, and Hytönen and Martikainen [22] further established a version of  $Tb$  theorem. Later, Hytönen et al. [20] and Bui and Duong [2], independently, introduced the atomic Hardy space  $H_{\text{atb}}^{1,q}(\mu)$  and proved that the dual space of  $H_{\text{atb}}^{1,q}(\mu)$  is  $\text{RBMO}(\mu)$ . Recently, Fu et al. [9, 10] established the boundedness of multilinear commutators of Calderón-Zygmund operators and commutators of generalized fractional integrals with  $\text{RBMO}(\mu)$ . The boundedness of commutators of multilinear singular integrals on Lebesgue spaces was obtained by Xie et al. [44]. In addition, Fu et al. [11] introduced a version of the atomic Hardy space

$\tilde{H}_{\text{atb},\rho}^{1,q,\gamma}(\mu) (\subset H_{\text{atb}}^{1,q}(\mu)$  and simply denoted by  $\tilde{H}^1(\mu)$ ; see Definition 2.11 below) and its corresponding dual space  $\widetilde{\text{RBMO}}(\mu) (\supset \text{RBMO}(\mu)$ ; see Definition 2.13 below) via the discrete coefficients  $\tilde{K}_{B,S}^{(\rho)}$ . Moreover, Hytönen and Martikainen [23] proved a non-homogeneous  $T1$  theorem for certain bi-parameter singular integral operators. Very recently, Fu et al. [12] partially established the theory of the Hardy space  $H^p$  with  $p \in (0, 1]$  on  $(\mathcal{X}, d, \mu)$ . Sawano et al. [33] presented an example showing that, if  $(\mathcal{X}, d, \mu)$  is not geometrically doubling, then Morrey spaces depend on the auxiliary parameters. More research on function spaces and the boundedness of various operators on metric measure spaces of non-homogeneous type can be found in [24, 21, 25, 27, 17, 1, 3, 26, 28, 4]. We refer the reader to the survey [45] and the monograph [46] for more developments on harmonic analysis in this setting.

The main purpose of this paper is to generalize the corresponding results in [15] to the present setting  $(\mathcal{X}, d, \mu)$ . Precisely, let  $T$  be a Calderón-Zygmund operator with kernel satisfying only the size condition and some Hörmander-type condition, and  $b \in \widetilde{\text{RBMO}}(\mu)$ . Under the assumption that  $T$  is bounded on  $L^2(\mu)$ , we obtain the boundedness of the commutator

$$T_b := bT - Tb,$$

generated by  $T$  and  $b$ , from the atomic Hardy space  $\tilde{H}^1(\mu)$  into the weak Lebesgue space  $L^{1,\infty}(\mu)$ . The boundedness of the commutator generated by the generalized fractional integral  $T_\alpha$  ( $\alpha \in (0, 1)$ ) and the  $\widetilde{\text{RBMO}}(\mu)$  function from  $\tilde{H}^1(\mu)$  into the weak Lebesgue space  $L^{1/(1-\alpha),\infty}(\mu)$  is also established. Moreover, by an interpolation theorem for sublinear operators, we also show that the commutator  $T_b$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ .

This paper is organized as follows. In Section 2, we first recall some necessary notation and notions, including the discrete coefficient  $\tilde{K}_{B,S}^{(\rho)}$  and its fundamental properties, the atomic Hardy space  $\tilde{H}_{\text{atb},\rho}^{1,q,\gamma}(\mu)$  (simply denoted by  $\tilde{H}^1(\mu)$ ) and the space  $\widetilde{\text{RBMO}}(\mu)$  with  $\tilde{K}_{B,S}^{(\rho)}$ , and the Calderón-Zygmund decomposition. We also establish an equivalent characterization and the John-Nirenberg inequality of  $\widetilde{\text{RBMO}}(\mu)$  (see, respectively, Lemma 2.15 and Proposition 2.16 below), whose proofs are similar to those of the corresponding known results of  $\text{RBMO}(\mu)$ , the details being omitted. Moreover, in this section, we find a useful property of the dominating function (see Lemma 2.3 below), which is of independent interest and is used in Section 3.

In Section 3, we establish the boundedness of the commutator  $T_b$  from  $\tilde{H}^1(\mu)$  into  $L^{1,\infty}(\mu)$  by borrowing some ideas from [15, Theorem 4.1] and applying Lemma 2.3.

In Section 4, we prove that the commutator, generated by the generalized fractional integral  $T_\alpha$  ( $\alpha \in (0, 1)$ ) and the  $\widetilde{\text{RBMO}}(\mu)$  function, is bounded from  $\tilde{H}^1(\mu)$  into  $L^{1/(1-\alpha),\infty}(\mu)$ . Recall that the fractional type of the discrete coefficient  $\tilde{K}_{B,S}^{(\rho)}$  is a useful tool in the study of commutators of fractional integrals in the setting of metric measure spaces with non-doubling measures or metric measure spaces of non-homogeneous type; see, for example, [6, 10]. However, in our proof, via the Minkowski integral inequality and the Fatou lemma, we do not need to use the fractional coefficient, which is a different approach to deal with commutators of fractional integrals.

Section 5 is devoted to the boundedness on  $L^p(\mu)$ , with  $p \in (1, \infty)$ , of the commutator

$T_b$ . To this end, we first establish an interpolation theorem for sublinear operators (see Theorem 5.5 below). Although the interpolation theorem is similar to [15, Theorem 3.1], its proof is different. Precisely, since it is not clear whether or not the operator  $M_r^\sharp \circ T_1$  compounded by the sharp maximal operator  $M_r^\sharp$  and the sublinear operator  $T_1$  is quasi-linear, the method used in the proof of [15, Theorem 3.1] might be problematic. To avoid this, in the below proof of Theorem 5.5, we borrow some ideas from the proof of [28, Theorem 1.6]. Then we establish a pointwise estimate for  $M_r^\sharp \circ T_b$ , which, together with the interpolation theorem, yields the desired conclusion.

Finally, we make some conventions on notation. Throughout this paper, we always denote by  $C$ ,  $\tilde{C}$ ,  $c$  or  $\tilde{c}$  a *positive constant* which is independent of the main parameters, but they may vary from line to line. *Constants with subscripts*, such as  $C_0$  and  $c_0$ , do not change in different occurrences. Furthermore, we use  $C_{(\alpha)}$  to denote a positive constant depending on the parameter  $\alpha$ . The expression  $Y \lesssim Z$  means that there exists a positive constant  $C$  such that  $Y \leq CZ$ . The expression  $A \sim B$  means that  $A \lesssim B \lesssim A$ . Let  $\mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ . For any ball  $B \subset \mathcal{X}$ , we denote its *center* and *radius*, respectively, by  $c_B$  and  $r_B$  and, moreover, for any  $\rho \in (0, \infty)$ , we denote the ball  $B(c_B, \rho r_B)$  by  $\rho B$ . Given any  $q \in (0, \infty)$ , let  $q' := q/(q-1)$  denote its *conjugate index*. Also, for any subset  $E \subset \mathcal{X}$ ,  $\chi_E$  denotes its *characteristic function*. For any  $f \in L^1_{\text{loc}}(\mu)$  and any measurable set  $E$  of  $\mathcal{X}$ ,  $m_E(f)$  denotes its mean over  $E$ , namely,

$$m_E(f) := \frac{1}{\mu(E)} \int_E f(x) d\mu(x).$$

For arbitrary  $a \in \mathbb{R}$ ,  $[a]$  denotes the largest integer smaller than or equal to  $a$ .

## 2 Preliminaries

In this section, we recall some necessary notions and notation, including the dominating function, the discrete coefficient  $\tilde{K}_{B,S}^{(\rho)}$ , the atomic Hardy space  $\tilde{H}_{\text{atb},\rho}^{1,q,\gamma}(\mu)$ , the space  $\widetilde{\text{RBMO}}(\mu)$  and the Calderón-Zygmund decomposition. We also give out a useful property of the dominating function.

The following notion of upper doubling metric measure spaces was originally introduced by Hytönen [19] (see also [21, 27]).

**Definition 2.1.** A metric measure space  $(\mathcal{X}, d, \mu)$  is said to be *upper doubling* if  $\mu$  is a Borel measure on  $\mathcal{X}$  and there exist a *dominating function*  $\lambda : \mathcal{X} \times (0, \infty) \rightarrow (0, \infty)$  and a positive constant  $C_{(\lambda)}$ , depending on  $\lambda$ , such that, for each  $x \in \mathcal{X}$ ,  $r \rightarrow \lambda(x, r)$  is non-decreasing and, for all  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ ,

$$(2.1) \quad \mu(B(x, r)) \leq \lambda(x, r) \leq C_{(\lambda)} \lambda(x, r/2).$$

**Remark 2.2.** (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function  $\lambda(x, r) := \mu(B(x, r))$  for all  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ . On the other hand, the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  with any Radon measure  $\mu$  as in (1.2) is also an upper doubling space by taking  $\lambda(x, r) := C_0 r^n$  for all  $x \in \mathbb{R}^d$  and  $r \in (0, \infty)$ .

- (ii) Let  $(\mathcal{X}, d, \mu)$  be upper doubling with  $\lambda$  being the dominating function on  $\mathcal{X} \times (0, \infty)$  as in Definition 2.1. It was proved in [20] that there exists another dominating function  $\tilde{\lambda}$  such that  $\tilde{\lambda} \leq \lambda$ ,  $C_{(\tilde{\lambda})} \leq C_{(\lambda)}$  and, for all  $x, y \in \mathcal{X}$  with  $d(x, y) \leq r$ ,

$$(2.2) \quad \tilde{\lambda}(x, r) \leq C_{(\tilde{\lambda})} \tilde{\lambda}(y, r).$$

- (iii) It was shown in [36] that the upper doubling condition is equivalent to the *weak growth condition*: there exist a dominating function  $\lambda : \mathcal{X} \times (0, \infty) \rightarrow (0, \infty)$ , with  $r \rightarrow \lambda(x, r)$  non-decreasing, positive constants  $C_{(\lambda)}$ , depending on  $\lambda$ , and  $\epsilon$  such that

- (iii)<sub>1</sub> for all  $r \in (0, \infty)$ ,  $t \in [0, r]$ ,  $x, y \in \mathcal{X}$  and  $d(x, y) \in [0, r]$ ,

$$|\lambda(y, r+t) - \lambda(x, r)| \leq C_{(\lambda)} \left[ \frac{d(x, y) + t}{r} \right]^\epsilon \lambda(x, r);$$

- (iii)<sub>2</sub> for all  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ ,  $\mu(B(x, r)) \leq \lambda(x, r)$ .

The following property of the dominating function  $\lambda$  is useful and of independent interest.

**Lemma 2.3.** *Let  $(\mathcal{X}, d, \mu)$  be an upper doubling space with dominating function  $\lambda$  satisfying (2.2) and ball  $B \subset \mathcal{X}$ . Then, for any  $x_1, x_2 \in B$  and  $y \in \mathcal{X} \setminus (kB)$  with  $k \in [2, \infty)$ , it holds true that  $\lambda(x_1, d(x_1, y)) \sim \lambda(x_2, d(x_2, y))$ .*

*Proof.* Without loss of generality, we may assume that  $d(x_1, y) \leq d(x_2, y)$ . By (2.2) and the fact that  $\lambda(x, r)$  is non-decreasing according to  $r$ , we have

$$\lambda(x_1, d(x_1, y)) \sim \lambda(y, d(x_1, y)) \leq \lambda(y, d(x_2, y)) \sim \lambda(x_2, d(x_2, y)).$$

Therefore, to prove Lemma 2.3, we only need to show that  $\lambda(x_2, d(x_2, y)) \lesssim \lambda(x_1, d(x_1, y))$ . Notice that, for  $x_1 \in B$  and  $y \in \mathcal{X} \setminus (kB)$ ,

$$d(x_1, y) \geq d(y, c_B) - d(x_1, c_B) > 2r_B - r_B = r_B.$$

It then follows that

$$d(x_2, y) \leq d(x_2, x_1) + d(x_1, y) < 2r_B + d(x_1, y) \leq 3d(x_1, y),$$

which, together with (2.2), the assumption that  $d(x_1, y) \leq d(x_2, y)$  and (2.1), implies that

$$\lambda(x_2, d(x_2, y)) \sim \lambda(y, d(x_2, y)) \lesssim \lambda(x_1, d(x_2, y)) \lesssim \lambda(x_1, 3d(x_1, y)) \lesssim \lambda(x_1, d(x_1, y)).$$

This finishes the proof of Lemma 2.3.  $\square$

The following definition of geometrically doubling is well known in analysis on metric spaces, which can be found in Coifman and Weiss [7, pp.66-67], and is also known as *metrically doubling* (see, for example, [14, p.81]). Moreover, spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [7, pp.66-68].

**Definition 2.4.** A metric space  $(\mathcal{X}, d)$  is said to be *geometrically doubling* if there exists some  $N_0 \in \mathbb{N}$  such that, for any ball  $B(x, r) \subset \mathcal{X}$  with  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ , there exists a finite ball covering  $\{B(x_i, r/2)\}_i$  of  $B(x, r)$  such that the cardinality of this covering is at most  $N_0$ .

**Remark 2.5.** Let  $(\mathcal{X}, d)$  be a metric space. In [19], Hytönen showed that the following statements are mutually equivalent:

- (i)  $(\mathcal{X}, d)$  is geometrically doubling;
- (ii) for any  $\epsilon \in (0, 1)$  and any ball  $B(x, r) \subset \mathcal{X}$  with  $x \in \mathcal{X}$  and  $r \in (0, \infty)$ , there exists a finite ball covering  $\{B(x_i, \epsilon r)\}_i$  of  $B(x, r)$  such that the cardinality of this covering is at most  $N_0 \epsilon^{-n_0}$ , here and hereafter,  $N_0$  is as in Definition 2.4 and  $n_0 := \log_2 N_0$ ;
- (iii) for every  $\epsilon \in (0, 1)$ , any ball  $B(x, r) \subset \mathcal{X}$  with  $x \in \mathcal{X}$  and  $r \in (0, \infty)$  contains at most  $N_0 \epsilon^{-n_0}$  centers of disjoint balls  $\{B(x_i, \epsilon r)\}_i$ ;
- (iv) there exists  $M \in \mathbb{N}$  such that any ball  $B(x, r) \subset \mathcal{X}$  with  $x \in \mathcal{X}$  and  $r \in (0, \infty)$  contains at most  $M$  centers  $\{x_i\}_i$  of disjoint balls  $\{B(x_i, r/4)\}_{i=1}^M$ .

A metric measure space  $(\mathcal{X}, d, \mu)$  is called a *metric measure space of non-homogeneous type* if  $(\mathcal{X}, d)$  is geometrically doubling and  $(\mathcal{X}, d, \mu)$  is upper doubling. Based on Remark 2.2(ii), from now on, we *always assume* that  $(\mathcal{X}, d, \mu)$  is a metric measure space of non-homogeneous type with the dominating function  $\lambda$  satisfying (2.2) and, for any two balls  $B, S \subset \mathcal{X}$ , if  $B = S$ , then  $c_B = c_S$  and  $r_B = r_S$ ; see [12, pp.314-315] for some details.

Although the measure doubling condition is not assumed uniformly for all balls in the metric measure space  $(\mathcal{X}, d, \mu)$  of non-homogeneous type, it was shown in [19] that there still exist many balls which have the following  $(\alpha, \beta)$ -doubling property.

**Definition 2.6.** Let  $\alpha, \beta \in (1, \infty)$ . A ball  $B \subset \mathcal{X}$  is said to be  $(\alpha, \beta)$ -doubling if  $\mu(\alpha B) \leq \beta \mu(B)$ .

To be precise, it was proved in [19, Lemma 3.2] that, if a metric measure space  $(\mathcal{X}, d, \mu)$  is upper doubling and  $\alpha, \beta \in (1, \infty)$  with  $\beta > [C(\lambda)]^{\log_2 \alpha} =: \alpha^\nu$ , then, for any ball  $B \subset \mathcal{X}$ , there exists some  $j \in \mathbb{Z}_+$  such that  $\alpha^j B$  is  $(\alpha, \beta)$ -doubling. Moreover, let  $(\mathcal{X}, d)$  be geometrically doubling,  $\beta > \alpha^{n_0}$  with  $n_0 := \log_2 N_0$  and  $\mu$  a Borel measure on  $\mathcal{X}$  which is finite on bounded sets. Hytönen [19, Lemma 3.3] also showed that, for  $\mu$ -almost every  $x \in \mathcal{X}$ , there exist arbitrary small  $(\alpha, \beta)$ -doubling balls centered at  $x$ . Furthermore, the radii of these balls may be chosen to be of the form  $\alpha^{-j} r$  for  $j \in \mathbb{N}$  and any preassigned number  $r \in (0, \infty)$ . Throughout this article, for any  $\alpha \in (1, \infty)$  and ball  $B$ , the *smallest*  $(\alpha, \beta_\alpha)$ -doubling ball of the form  $\alpha^j B$  with  $j \in \mathbb{Z}_+$  is denoted by  $\tilde{B}^\alpha$ , where

$$(2.3) \quad \beta_\alpha := \alpha^{3(\max\{n_0, \nu\})} + [\max\{5\alpha, 30\}]^{n_0} + [\max\{3\alpha, 30\}]^\nu.$$

Also, for any ball  $B$  of  $\mathcal{X}$ , we denote by  $\tilde{B}$  the smallest  $(6, \beta_6)$ -doubling cube of the form  $6^j B$  with  $j \in \mathbb{Z}_+$ , especially, throughout this paper.

The following discrete coefficient  $\tilde{K}_{B,S}^{(\rho)}$  was first introduced by Bui and Duong [2] as analogous of the quantity introduced by Tolsa [38] (see also [39, 41]) in the setting of non-doubling measures; see also [11, 12].



**Definition 2.7.** For any  $\rho \in (1, \infty)$  and any two balls  $B \subset S \subset \mathcal{X}$ , let

$$\tilde{K}_{B,S}^{(\rho)} := 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N_{B,S}^{(\rho)}} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)},$$

where  $N_{B,S}^{(\rho)}$  is the *smallest integer* satisfying  $\rho^{N_{B,S}^{(\rho)}} r_B \geq r_S$ .

**Remark 2.8.** (i) By a change of variables and (2.1), we easily conclude that

$$\tilde{K}_{B,S}^{(\rho)} \sim 1 + \sum_{k=1}^{N_{B,S}^{(\rho)} + \lfloor \log_\rho 2 \rfloor + 1} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)},$$

where the implicit equivalent positive constants are independent of balls  $B \subset S \subset \mathcal{X}$ , but depend on  $\rho$ .

(ii) A continuous version,  $K_{B,S}$ , of the coefficient in Definition 2.7 was introduced in [19] and [20] as follows. For any two balls  $B \subset S \subset \mathcal{X}$ , let

$$K_{B,S} := 1 + \int_{(2S) \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x).$$

It was proved in [20] that  $K_{B,S}$  has all properties similar to those for  $\tilde{K}_{B,S}^{(\rho)}$  as in Lemma 2.9 below. Unfortunately,  $K_{B,S}$  and  $\tilde{K}_{B,S}^{(\rho)}$  are usually not equivalent, but, for  $(\mathbb{R}^d, |\cdot|, \mu)$  with  $\mu$  as in (1.2),

$$(2.4) \quad K_{B,S} \sim \tilde{K}_{B,S}^{(\rho)}$$

with implicit equivalent positive constants independent of  $B$  and  $S$ ; see [11] for more details on this.

The following useful properties of  $\tilde{K}_{B,S}^{(\rho)}$  were proved in [12].

**Lemma 2.9.** *Let  $(\mathcal{X}, d, \mu)$  be a metric measure space of non-homogeneous type.*

(i) *For any  $\rho \in (1, \infty)$ , there exists a positive constant  $C_{(\rho)}$ , depending on  $\rho$ , such that, for all balls  $B \subset R \subset S$ ,  $\tilde{K}_{B,R}^{(\rho)} \leq C_{(\rho)} \tilde{K}_{B,S}^{(\rho)}$ .*

(ii) *For any  $\alpha \in [1, \infty)$  and  $\rho \in (1, \infty)$ , there exists a positive constant  $C_{(\alpha, \rho)}$ , depending on  $\alpha$  and  $\rho$ , such that, for all balls  $B \subset S$  with  $r_S \leq \alpha r_B$ ,  $\tilde{K}_{B,S}^{(\rho)} \leq C_{(\alpha, \rho)}$ .*

(iii) *For any  $\rho \in (1, \infty)$ , there exists a positive constant  $C_{(\rho, \nu)}$ , depending on  $\rho$  and  $\nu$ , such that, for all balls  $B$ ,  $\tilde{K}_{B, \tilde{B}^\nu}^{(\rho)} \leq C_{(\rho, \nu)}$ . Moreover, letting  $\alpha, \beta \in (1, \infty)$ ,  $B \subset S$  be any two concentric balls such that there exists no  $(\alpha, \beta)$ -doubling ball in the form of  $\alpha^k B$  with  $k \in \mathbb{N}$ , satisfying  $B \subset \alpha^k B \subset S$ , then there exists a positive constant  $C_{(\alpha, \beta, \nu)}$ , depending on  $\alpha, \beta$  and  $\nu$ , such that  $\tilde{K}_{B,S}^{(\rho)} \leq C_{(\alpha, \beta, \nu)}$ .*

(iv) For any  $\rho \in (1, \infty)$ , there exists a positive constant  $c_{(\rho, \nu)}$ , depending on  $\rho$  and  $\nu$ , such that, for all balls  $B \subset R \subset S$ ,

$$\tilde{K}_{B,S}^{(\rho)} \leq \tilde{K}_{B,R}^{(\rho)} + c_{(\rho, \nu)} \tilde{K}_{R,S}^{(\rho)}.$$

(v) For any  $\rho \in (1, \infty)$ , there exists a positive constant  $\tilde{c}_{(\rho, \nu)}$ , depending on  $\rho$  and  $\nu$ , such that, for all balls  $B \subset R \subset S$ ,  $\tilde{K}_{R,S}^{(\rho)} \leq \tilde{c}_{(\rho, \nu)} \tilde{K}_{B,S}^{(\rho)}$ .

**Lemma 2.10.** Let  $(\mathcal{X}, d, \mu)$  be a metric measure space of non-homogeneous type and  $\rho_1, \rho_2 \in (1, \infty)$ . Then there exist positive constants  $c_{(\rho_1, \rho_2, \nu)}$  and  $C_{(\rho_1, \rho_2, \nu)}$ , depending on  $\rho_1, \rho_2$  and  $\nu$ , such that, for all balls  $B \subset S$ ,

$$c_{(\rho_1, \rho_2, \nu)} \tilde{K}_{B,S}^{(\rho_1)} \leq \tilde{K}_{B,S}^{(\rho_2)} \leq C_{(\rho_1, \rho_2, \nu)} \tilde{K}_{B,S}^{(\rho_1)}.$$

Now we recall the atomic Hardy space  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  and its dual space  $\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)$  associated with  $\tilde{K}_{B,S}^{(\rho)}$ , which were first introduced by Fu et al. [11].

**Definition 2.11.** Let  $\rho \in (1, \infty)$ ,  $q \in (1, \infty]$  and  $\gamma \in [1, \infty)$ . A function  $b \in L^1(\mu)$  is called a  $(q, \gamma, \rho)_\lambda$ -atomic block if

- (i) there exists a ball  $B$  such that  $\text{supp } b \subset B$ ;
- (ii)  $\int_{\mathcal{X}} b(x) d\mu(x) = 0$ ;
- (iii) for any  $j \in \{1, 2\}$ , there exist a function  $a_j$  supported on a ball  $B_j \subset B$  and a number  $\lambda_j \in \mathbb{C}$  such that  $b = \lambda_1 a_1 + \lambda_2 a_2$  and

$$\|a_j\|_{L^q(\mu)} \leq [\mu(\rho B_j)]^{1/q-1} \left[ \tilde{K}_{B_j, B}^{(\rho)} \right]^{-\gamma}.$$

Moreover, let

$$|b|_{\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)} := |\lambda_1| + |\lambda_2|.$$

A function  $f \in L^1(\mu)$  is said to belong to the atomic Hardy space  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  if there exist  $(q, \gamma, \rho)_\lambda$ -atomic blocks  $\{b_i\}_{i=1}^\infty$  such that  $f = \sum_{i=1}^\infty b_i$  in  $L^1(\mu)$  and

$$\sum_{i=1}^\infty |b_i|_{\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)} < \infty.$$

The  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  norm of  $f$  is defined by

$$\|f\|_{\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)} := \inf \left\{ \sum_{i=1}^\infty |b_i|_{\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of  $f$  as above.



**Remark 2.12.** (i) When  $(\mathcal{X}, d, \mu) = (\mathbb{R}^d, |\cdot|, \mu)$  with  $\mu$  as in (1.2), by (2.4), we see that  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  becomes the atomic Hardy space  $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  in [38] for  $\gamma = 1$  and in [15] for  $\gamma \in (1, \infty)$ . For general metric measure spaces of non-homogeneous type, if we replace  $\tilde{K}_{B, S}^{(\rho)}$  by  $K_{B, S}$  in Definition 2.11, then  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  becomes the atomic Hardy space  $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  in [2, 20]. Obviously, for  $\rho \in (1, \infty)$ ,  $q \in (1, \infty]$  and  $\gamma \in [1, \infty)$ , we always have

$$\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu) \subset H_{\text{atb}, \rho}^{1, q, \gamma}(\mu).$$

(ii) It was pointed out by Fu et al. [11] that, for each  $q \in (1, \infty]$ , the atomic Hardy space  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  is independent of the choices of  $\rho$  and  $\gamma$  and that, for all  $q \in (1, \infty)$ , the spaces  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  and  $\tilde{H}_{\text{atb}, \rho}^{1, \infty, \gamma}(\mu)$  coincide with equivalent norms. Thus, in what follows, we denote  $\tilde{H}_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$  simply by  $\tilde{H}^1(\mu)$ .

**Definition 2.13.** Let  $\rho \in (1, \infty)$  and  $\gamma \in [1, \infty)$ . A function  $f \in L_{\text{loc}}^1(\mu)$  is said to be in the space  $\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)$  if there exist a positive constant  $\tilde{C}$  and, for any ball  $B \subset \mathcal{X}$ , a number  $f_B$  such that

$$(2.5) \quad \frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| d\mu(x) \leq \tilde{C}$$

and, for any two balls  $B$  and  $B_1$  such that  $B \subset B_1$ ,

$$(2.6) \quad |f_B - f_{B_1}| \leq \tilde{C} \left[ \tilde{K}_{B, B_1}^{(\rho)} \right]^\gamma.$$

The infimum of the positive constant  $\tilde{C}$  satisfying both (2.5) and (2.6) is defined to be the  $\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)$  norm of  $f$  and denoted by  $\|f\|_{\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)}$ .

**Remark 2.14.** (i) It was pointed out by Fu et al. [11] that the space  $\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)$  is independent of  $\rho \in (1, \infty)$  and  $\gamma \in [1, \infty)$ . In what follows, we denote  $\widetilde{\text{RBMO}}_{\rho, \gamma}(\mu)$  simply by  $\widetilde{\text{RBMO}}(\mu)$ .

(ii) When  $(\mathcal{X}, d, \mu) = (\mathbb{R}^d, |\cdot|, \mu)$  with  $\mu$  as in (1.2), by (2.4), we see that  $\widetilde{\text{RBMO}}(\mu)$  becomes the regularized BMO( $\mu$ ) space, RBMO( $\mu$ ), introduced in [38] for  $\gamma = 1$  and in [15] for  $\gamma \in (1, \infty)$ . For general metric measure spaces of non-homogeneous type, if we replace  $\tilde{K}_{B, S}^{(\rho)}$  by  $K_{B, S}$  in Definition 2.13, then  $\widetilde{\text{RBMO}}(\mu)$  becomes the space RBMO( $\mu$ ) in [19]. Obviously, for  $\rho \in (1, \infty)$  and  $\gamma \in [1, \infty)$ ,  $\text{RBMO}(\mu) \subset \widetilde{\text{RBMO}}(\mu)$ . However, it is still unclear whether we always have  $\text{RBMO}(\mu) = \widetilde{\text{RBMO}}(\mu)$  or not.

(iii) Let  $\rho \in (1, \infty)$ ,  $p \in (1, \infty]$  and  $\gamma \in [1, \infty)$ . It was pointed out by Fu et al. [11] that  $[\tilde{H}_{\text{atb}, \rho}^{1, p, \gamma}(\mu)]^* = \widetilde{\text{RBMO}}(\mu)$ .

By some arguments similar to those used in the proofs of [20, Proposition 2.10] and [24, Lemma 3.2], we obtain the following equivalent characterization of the space  $\widetilde{\text{RBMO}}(\mu)$ , the details being omitted.

**Lemma 2.15.** *Let  $\eta, \rho \in (1, \infty)$ , and  $\beta_\rho$  be as in (2.4). For  $f \in L^1_{loc}(\mu)$ , the following statements are equivalent:*

- (i)  $f \in \widetilde{\text{RBMO}}(\mu)$ ;
- (ii) *there exists a positive constant  $C$  such that, for all balls  $B$ ,*

$$(2.7) \quad \frac{1}{\mu(\eta B)} \int_B |f(x) - m_{\widetilde{B}^\rho}(f)| d\mu(x) \leq C$$

and, for all  $(\rho, \beta_\rho)$ -doubling balls  $B \subset S$ ,

$$(2.8) \quad |m_B(f) - m_S(f)| \leq C \widetilde{K}_{B,S}^{(\rho)}.$$

Moreover, the infimum of the above constant  $C$  is equivalent to  $\|f\|_{\widetilde{\text{RBMO}}(\mu)}$ .

By an argument completely analogous to that used in the proof of [19, Proposition 6.1], we obtain the following John-Nirenberg inequality for  $\widetilde{\text{RBMO}}(\mu)$ , the details being omitted.

**Proposition 2.16.** *Let  $(\mathcal{X}, d, \mu)$  be a metric measure space of non-homogeneous type. Then, for every  $\rho \in (0, \infty)$ , there exists a positive constant  $c$  such that, for all  $f \in \widetilde{\text{RBMO}}(\mu)$ , balls  $B_0$  and  $t \in (0, \infty)$ ,*

$$\mu(\{x \in B_0 : |f(x) - f_{B_0}| > t\}) \leq 2\mu(\rho B_0) e^{-ct/\|f\|_{\widetilde{\text{RBMO}}(\mu)}},$$

where  $f_{B_0}$  is as in Definition 2.13 with  $B$  replaced by  $B_0$ .

**Corollary 2.17.** *Let  $(\mathcal{X}, d, \mu)$  be a metric measure space of non-homogeneous type. Then, for every  $\rho \in (1, \infty)$  and  $p \in [1, \infty)$ , there exists a constant  $C$  such that, for all  $f \in \widetilde{\text{RBMO}}(\mu)$  and balls  $B$ ,*

$$\left[ \frac{1}{\mu(\rho B)} \int_B |f(x) - f_B|^p d\mu(x) \right]^{1/p} \leq C \|f\|_{\widetilde{\text{RBMO}}(\mu)},$$

where  $f_B$  is as in Definition 2.13.

At the end of this section, we establish the following Calderón-Zygmund decomposition analogous to [2, Theorem 6.3] and its proof is also analogous to that of [2, Theorem 6.3], the details being omitted. Let  $\gamma_0$  be a fixed positive constant satisfying that  $\gamma_0 > \max\{C_{(\lambda)}^{3 \log_2 6}, 6^{3n_0}\}$ , where  $C_{(\lambda)}$  is as in (2.1) and  $n_0$  is as in Remark 2.5(ii).

**Lemma 2.18.** *Let  $p \in [1, \infty)$ ,  $f \in L^p(\mu)$  and  $t \in (0, \infty)$  ( $t > (\gamma_0)^{1/p} \|f\|_{L^p(\mu)} / [\mu(\mathcal{X})]^{1/p}$  when  $\mu(\mathcal{X}) < \infty$ ). Then the following hold true.*

- (i) *There exists an almost disjoint family  $\{6B_j\}_j$  of balls such that  $\{B_j\}_j$  is pairwise disjoint,*

$$\frac{1}{\mu(6^2 B_j)} \int_{B_j} |f(x)|^p d\mu(x) > \frac{t^p}{\gamma_0} \quad \text{for all } j,$$

$$\frac{1}{\mu(6^2\eta B_j)} \int_{\eta B_j} |f(x)|^p d\mu(x) \leq \frac{t^p}{\gamma_0} \quad \text{for all } j \text{ and all } \eta \in (2, \infty),$$

and

$$|f(x)| \leq t \quad \text{for } \mu - \text{almost every } x \in \mathcal{X} \setminus (\cup_j 6B_j).$$

(ii) For each  $j$ , let  $S_j$  be a  $(3 \times 6^2, C_\lambda^{\log_2(3 \times 6^2)+1})$ -doubling ball of the family  $\{(3 \times 6^2)^k B_j\}_{k \in \mathbb{N}}$  and  $\omega_j := \chi_{6B_j} / (\sum_k \chi_{6B_k})$ . Then there exists a family  $\{\varphi_j\}_j$  of functions such that, for each  $j$ ,  $\text{supp}(\varphi_j) \subset S_j$ ,  $\varphi_j$  has a constant sign on  $S_j$ ,

$$\int_{\mathcal{X}} \varphi_j(x) d\mu(x) = \int_{6B_j} f(x) \omega_j(x) d\mu(x),$$

$$\sum_j |\varphi_j(x)| \leq \gamma t \quad \text{for } \mu - \text{almost every } x \in \mathcal{X},$$

where  $\gamma$  is some positive constant, depending only on  $(\mathcal{X}, \mu)$ , and there exists a positive constant  $C$ , independent of  $f$ ,  $t$  and  $j$ , such that, when  $p = 1$ , it holds true that

$$\|\varphi_j\|_{L^\infty(\mu)} \mu(S_j) \leq C \int_{\mathcal{X}} |f(x) \omega_j(x)| d\mu(x)$$

and, when  $p \in (1, \infty)$ , it holds true that

$$\left[ \int_{S_j} |\varphi_j(x)|^p d\mu(x) \right]^{1/p} [\mu(S_j)]^{1/p'} \leq \frac{C}{t^{p-1}} \int_{\mathcal{X}} |f(x) \omega_j(x)|^p d\mu(x).$$

(iii) For  $p \in (1, \infty)$ , if choosing  $S_j$  in (ii) to be the smallest  $(3 \times 6^2, C_\lambda^{\log_2(3 \times 6^2)+1})$ -doubling ball of the family  $\{(3 \times 6^2)^k B_j\}_{k \in \mathbb{N}}$ , then  $h := \sum_j (f \omega_j - \varphi_j) \in \tilde{H}^1(\mu)$  and there exists a positive constant  $C$ , independent of  $f$  and  $t$ , such that

$$\|h\|_{\tilde{H}^1(\mu)} \leq \frac{C}{t^{p-1}} \|f\|_{L^p(\mu)}^p.$$

### 3 Boundedness of Commutators $T_b$ on $\tilde{H}^1(\mu)$

In this section, we consider the boundedness from  $\tilde{H}^1(\mu)$  into  $L^{1,\infty}(\mu)$  of the commutator generated by the  $\widetilde{\text{RBM}}\mathcal{O}(\mu)$  function and the Calderón-Zygmund operator with kernel satisfying only the size condition and some Hörmander-type condition.

To be precise, let  $K$  be a  $\mu$ -locally integrable function on  $\{\mathcal{X} \times \mathcal{X}\} \setminus \{(x, x) : x \in \mathcal{X}\}$  satisfying the *size condition* that there exists a positive constant  $C$  such that, for all  $x, y \in \mathcal{X}$  with  $x \neq y$ ,

$$(3.1) \quad |K(x, y)| \leq C \frac{1}{\lambda(x, d(x, y))},$$

and the *Hörmander-type condition* that there exists a positive constant  $C$  such that, for any  $R \in (0, \infty)$  and  $y, y' \in \mathcal{X}$  with  $d(y, y') < R$ ,

$$(3.2) \quad \sum_{l=1}^{\infty} l \int_{6^l R < d(x, y) \leq 6^{l+1} R} [|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)|] d\mu(x) \leq C.$$

A linear operator  $T$  is called a *Calderón-Zygmund operator* with kernel  $K$  satisfying (3.1) and (3.2) if, for all  $f \in L_b^\infty(\mu) := \{f \in L^\infty(\mu) : \text{supp}(f) \text{ is bounded}\}$ ,

$$(3.3) \quad Tf(x) := \int_{\mathcal{X}} K(x, y) f(y) d\mu(y), \quad x \notin \text{supp}(f).$$

Let  $b \in \widetilde{\text{RBMO}}(\mu)$  and  $T$  be a Calderón-Zygmund operator defined above. The commutator  $T_b$ , generated by  $b$  and  $T$ , is defined by setting, for any suitable function  $f$ ,

$$(3.4) \quad T_b f := bTf - T(bf).$$

Now we state the main result of this section as follows.

**Theorem 3.1.** *Let  $b \in \widetilde{\text{RBMO}}(\mu)$ . Assume that the Calderón-Zygmund operator  $T$ , defined by (3.3) associated with kernel  $K$  satisfying (3.1) and (3.2), is bounded on  $L^2(\mu)$ . Then the commutator  $T_b$  defined by (3.4) is bounded from  $\tilde{H}^1(\mu)$  into  $L^{1,\infty}(\mu)$ , that is, there exists a positive constant  $C$  such that, for all  $t \in (0, \infty)$  and all functions  $f \in \tilde{H}^1(\mu)$ ,*

$$\mu(\{x \in \mathcal{X} : |T_b f(x)| > t\}) \leq C \|b\|_{\widetilde{\text{RBMO}}(\mu)} t^{-1} \|f\|_{\tilde{H}^1(\mu)}.$$

To prove Theorem 3.1, we need the following two lemmas.

**Lemma 3.2.** *Let  $T$  be a Calderón-Zygmund operator defined by (3.3) associated with kernel  $K$  satisfying (3.1) and (3.2). Assume that  $T$  is bounded on  $L^2(\mu)$ . Then*

- (i)  $T$  is bounded from  $L^1(\mu)$  into  $L^{1,\infty}(\mu)$ ;
- (ii)  $T$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ .

*Proof.* The proof of (i) is similar to that of (i)  $\implies$  (ii) of [28, Theorem 1.6], the details being omitted. By (i), together with the Marcinkiewicz interpolation theorem and a standard duality, we then obtain the desired result of (ii), which completes the proof of Lemma 3.2.  $\square$

The following generalized Hölder inequality is a special case of [9, Lemma 4.1] (see also [16, pp. 246-247] in the setting of  $\mathbb{R}^d$  with  $\mu$  as in (1.2) and [31, Lemmas 2.2 and 2.3] for the setting of  $\mathbb{R}^d$  with  $\mu$  being the  $d$ -dimensional Lebesgue measure).

**Lemma 3.3.** *There exists a positive constant  $C$  such that, for all locally integrable functions  $f$  and  $g$ , and all balls  $B$ ,*

$$(3.5) \quad \frac{1}{\mu(2B)} \int_B |f(x)g(x)| d\mu(x) \leq C \|g\|_{\exp L(\mu), B} \|f\|_{L \log L(\mu), B},$$

where

$$\|f\|_{L \log L(\mu), B} := \inf \left\{ s \in (0, \infty) : \frac{1}{\mu(2B)} \int_B \frac{|f(x)|}{s} \log \left( 2 + \frac{|f(x)|}{s} \right) d\mu(x) \leq 1 \right\}$$

and

$$\|f\|_{\exp L(\mu), B} := \inf \left\{ s \in (0, \infty) : \frac{1}{\mu(2B)} \int_B \exp \left( \frac{|f(x)|}{s} \right) d\mu(x) \leq 2 \right\}.$$

Now we can show Theorem 3.1 as follows.

*Proof of Theorem 3.1.* For each fixed  $f \in \tilde{H}^1(\mu)$ , by Definition 2.11, we have a decomposition  $f = \sum_{j=1}^{\infty} h_j$ , where, for any  $j \in \mathbb{N}$ ,  $h_j$  is an  $(\infty, 2, 12)_\lambda$ -atomic block,  $\text{supp } h_j \subset S_j$ ,  $S_j$  is a ball of  $\mathcal{X}$ , and

$$\sum_{j=1}^{\infty} |h_j|_{\tilde{H}_{\text{atb}, 12}^{1, \infty, 2}(\mu)} \leq 2 \|f\|_{\tilde{H}^1(\mu)}.$$

Moreover, for each fixed  $j$ , we can further decompose  $h_j$  as  $h_j = r_{j,1}a_{j,1} + r_{j,2}a_{j,2}$ , where, for any  $i \in \{1, 2\}$ ,  $r_{j,i} \in \mathbb{C}$ ,  $a_{j,i}$  is a bounded function supported on some ball  $B_{j,i} \subset S_j$  satisfying

$$\|a_{j,i}\|_{L^\infty(\mu)} \leq \left[ \mu(12B_{j,i}) \left\{ \tilde{K}_{B_{j,i}, S_j}^{(12)} \right\}^2 \right]^{-1}$$

and  $|h_j|_{\tilde{H}_{\text{atb}, 12}^{1, \infty, 2}(\mu)} = |r_{j,1}| + |r_{j,2}|$ . By Lemma 2.10, we further conclude that, for any  $j$  and any  $i \in \{1, 2\}$ ,

$$(3.6) \quad \|a_{j,i}\|_{L^\infty(\mu)} \lesssim \left[ \mu(12B_{j,i}) \left\{ \tilde{K}_{B_{j,i}, S_j}^{(6)} \right\}^2 \right]^{-1}.$$

Write

$$T_b f = \sum_{j=1}^{\infty} [b - m_{\tilde{S}_j}(b)] T h_j + T \left( \sum_{j=1}^{\infty} [m_{\tilde{S}_j}(b) - b(\cdot)] h_j \right) =: T_b^{\text{I}} f + T_b^{\text{II}} f.$$

By Lemma 3.2, we know that  $T$  is bounded from  $L^1(\mu)$  into  $L^{1, \infty}(\mu)$ . It then follows that

$$\begin{aligned} (3.7) \quad \mu(\{x \in \mathcal{X} : |T_b^{\text{II}} f(x)| > t\}) &\lesssim \frac{1}{t} \sum_{j=1}^{\infty} \int_{S_j} |b(x) - m_{\tilde{S}_j}(b)| |h_j(x)| d\mu(x) \\ &\lesssim \frac{1}{t} \sum_{j=1}^{\infty} \left[ |r_{j,1}| \int_{B_{j,1}} |b(x) - m_{\tilde{S}_j}(b)| |a_{j,1}(x)| d\mu(x) \right. \\ &\quad \left. + |r_{j,2}| \int_{B_{j,2}} |b(x) - m_{\tilde{S}_j}(b)| |a_{j,2}(x)| d\mu(x) \right] \end{aligned}$$

$$=: \frac{1}{t} \sum_{j=1}^{\infty} (E + F).$$

By (2.8) and Lemma 2.9, we have

$$\left| m_{\tilde{S}_j}(b) - m_{\tilde{B}_{j,1}}(b) \right| \lesssim \tilde{K}_{\tilde{B}_{j,1}, \tilde{S}_j}^{(6)} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \lesssim \tilde{K}_{B_{j,1}, S_j}^{(6)} \|b\|_{\widetilde{\text{RBMO}}(\mu)},$$

which, together with (2.7) and (3.6), leads to

$$\begin{aligned} E &\lesssim |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \left[ \int_{B_{j,1}} \left| b(x) - m_{\tilde{B}_{j,1}}(b) \right| d\mu(x) + \left| m_{\tilde{S}_j}(b) - m_{\tilde{B}_{j,1}}(b) \right| \mu(B_{j,1}) \right] \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \left[ \mu(12B_{j,1}) \left\{ \tilde{K}_{B_{j,1}, S_j}^{(6)} \right\}^2 \right]^{-1} \left[ \mu(2B_{j,1}) + \tilde{K}_{B_{j,1}, S_j}^{(6)} \mu(B_{j,1}) \right] \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}|. \end{aligned}$$

Similarly,

$$F \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,2}|.$$

Combining E and F, we conclude that

$$(3.8) \quad \mu(\{x \in \mathcal{X} : |T_b^{\text{II}} f(x)| > t\}) \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} t^{-1} \|f\|_{\tilde{H}^1(\mu)}.$$

Now we turn to estimate  $T_b^{\text{I}} f$ . Write

$$\begin{aligned} \mu(\{x \in \mathcal{X} : |T_b^{\text{I}} f(x)| > t\}) &\leq t^{-1} \sum_{j=1}^{\infty} \int_{6S_j} \left| b(x) - m_{\tilde{S}_j}(b) \right| |Th_j(x)| d\mu(x) \\ &\quad + t^{-1} \sum_{j=1}^{\infty} \int_{\mathcal{X} \setminus 6S_j} \cdots =: G + H. \end{aligned}$$

We first estimate G. For each fixed  $j$ , write

$$\begin{aligned} \int_{6S_j} \left| b(x) - m_{\tilde{S}_j}(b) \right| |Th_j(x)| d\mu(x) &\leq |r_{j,1}| \int_{6S_j} \left| b(x) - m_{\tilde{S}_j}(b) \right| |Ta_{j,1}(x)| d\mu(x) \\ &\quad + |r_{j,2}| \int_{6S_j} \left| b(x) - m_{\tilde{S}_j}(b) \right| |Ta_{j,2}(x)| d\mu(x) \\ &=: L_{j,1} + L_{j,2}. \end{aligned}$$

Since the two terms  $L_{j,1}$  and  $L_{j,2}$  can be estimated in a similar way, we only deal with  $L_{j,1}$ . Write

$$L_{j,1} \leq |r_{j,1}| \int_{6S_j \setminus 6B_{j,1}} \left| b(x) - m_{\tilde{S}_j}(b) \right| |Ta_{j,1}(x)| d\mu(x)$$

$$\begin{aligned}
& + |r_{j,1}| \int_{6B_{j,1}} \left| b(x) - m_{\widetilde{6B_{j,1}}}(b) \right| |Ta_{j,1}(x)| d\mu(x) \\
& + |r_{j,1}| \left| m_{\widetilde{6B_{j,1}}}(b) - m_{\widetilde{S_j}}(b) \right| \int_{6B_{j,1}} |Ta_{j,1}(x)| d\mu(x) =: U_j + V_j + W_j.
\end{aligned}$$

The Hölder inequality, together with (2.8), Lemma 2.9, the boundedness of  $T$  on  $L^2(\mu)$  and (3.6), implies that

$$\begin{aligned}
W_j & \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \widetilde{K}_{\widetilde{6B_{j,1}}, \widetilde{S_j}}^{(6)} \|Ta_{j,1}\|_{L^2(\mu)} [\mu(6B_{j,1})]^{\frac{1}{2}} \\
& \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \widetilde{K}_{B_{j,1}, S_j}^{(6)} \|a_{j,1}\|_{L^2(\mu)} [\mu(6B_{j,1})]^{\frac{1}{2}} \\
& \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \widetilde{K}_{B_{j,1}, S_j}^{(6)} \mu(12B_{j,1}) \|a_{j,1}\|_{L^\infty(\mu)} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}|.
\end{aligned}$$

From the Hölder inequality, Corollary 2.17, the boundedness of  $T$  on  $L^2(\mu)$  and (3.6), we deduce that

$$\begin{aligned}
V_j & \lesssim |r_{j,1}| \left[ \int_{6B_{j,1}} \left| b(x) - m_{\widetilde{6B_{j,1}}}(b) \right|^2 d\mu(x) \right]^{\frac{1}{2}} \|Ta_{j,1}\|_{L^2(\mu)} \\
& \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| [\mu(12B_{j,1})]^{\frac{1}{2}} \|a_{j,1}\|_{L^2(\mu)} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}|.
\end{aligned}$$

To estimate  $U_j$ , we first observe that, for  $x \notin 6B_{j,1}$  and  $y \in B_{j,1}$ ,  $d(x, y) \sim d(x, c_{B_{j,1}})$ . It then follows from (3.1) that, for  $x \notin 6B_{j,1}$ ,

$$\begin{aligned}
|Ta_{j,1}(x)| & = \left| \int_{\mathcal{X}} K(x, y) a_{j,1}(y) d\mu(y) \right| \leq \int_{B_{j,1}} |K(x, y)| |a_{j,1}(y)| d\mu(y) \\
& \lesssim \frac{\|a_{j,1}\|_{L^1(\mu)}}{\lambda(x, d(x, c_{B_{j,1}}))} \lesssim \frac{\|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1})}{\lambda(x, d(x, c_{B_{j,1}}))}.
\end{aligned}$$

Let  $N_1 := N_{6B_{j,1}, 6S_j}^{(6)} + [\log_6 2] + 1$ . A straightforward computation, via the above estimate, (2.7), (2.8), Lemma 2.9, (2.2) and (3.6), shows that

$$\begin{aligned}
U_j & \lesssim |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1}) \int_{6S_j \setminus 6B_{j,1}} \frac{|b(x) - m_{\widetilde{S_j}}(b)|}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x) \\
& \lesssim |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1}) \sum_{k=1}^{N_1} \left[ \int_{6^{k+1}B_{j,1} \setminus 6^k B_{j,1}} \frac{|b(x) - m_{\widetilde{6^{k+1}B_{j,1}}}(b)|}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x) \right. \\
& \quad \left. + \left| m_{\widetilde{6^{k+1}B_{j,1}}}(b) - m_{\widetilde{S_j}}(b) \right| \int_{6^{k+1}B_{j,1} \setminus 6^k B_{j,1}} \frac{1}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x) \right] \\
& \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1}) \\
& \quad \times \sum_{k=1}^{N_1} \left[ \frac{\mu(6^{k+2}B_{j,1})}{\lambda(c_{B_{j,1}}, 6^{k+2}r_{B_{j,1}})} + \widetilde{K}_{B_{j,1}, S_j}^{(6)} \frac{\mu(6^{k+1}B_{j,1})}{\lambda(c_{B_{j,1}}, 6^{k+1}r_{B_{j,1}})} \right]
\end{aligned}$$



$$\begin{aligned}
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1}) \tilde{K}_{B_{j,1}, S_j}^{(6)} \left[ 1 + \sum_{k=1}^{N_1} \frac{\mu(6^k B_{j,1})}{\lambda(c_{B_{j,1}}, 6^k r_{B_{j,1}})} \right] \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}| \|a_{j,1}\|_{L^\infty(\mu)} \mu(B_{j,1}) \left[ \tilde{K}_{B_{j,1}, S_j}^{(6)} \right]^2 \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}|.
\end{aligned}$$

Combining the estimates for  $U_j$ ,  $V_j$  and  $W_j$ , we obtain

$$L_{j,1} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} |r_{j,1}|,$$

which further implies that

$$G \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{j=1}^{\infty} |h_j|_{\tilde{H}_{\text{atb}, 12}^{1, \infty, 2}(\mu)} \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{\tilde{H}^1(\mu)}.$$

It remains to estimate  $H$ . The vanishing moment of  $h_j$ , together with the Fubini theorem, implies that

$$\begin{aligned}
H &= t^{-1} \sum_{j=1}^{\infty} \int_{\mathcal{X} \setminus 6S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right| \left| \int_{S_j} [K(x, y) - K(x, c_{S_j})] h_j(y) d\mu(y) \right| d\mu(x) \\
&\leq t^{-1} \sum_{j=1}^{\infty} \int_{S_j} |h_j(y)| \int_{\mathcal{X} \setminus 6S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right| |K(x, y) - K(x, c_{S_j})| d\mu(x) d\mu(y).
\end{aligned}$$

For each fixed  $j$ , write

$$\begin{aligned}
&\int_{\mathcal{X} \setminus 6S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right| |K(x, y) - K(x, c_{S_j})| d\mu(x) \\
&\leq \sum_{k=1}^{\infty} \int_{6^{k+1}S_j \setminus 6^k S_j} \left| b(x) - m_{\widetilde{6^{k+1}S_j}}(b) \right| |K(x, y) - K(x, c_{S_j})| d\mu(x) \\
&\quad + \sum_{k=1}^{\infty} \int_{6^{k+1}S_j \setminus 6^k S_j} \left| m_{\widetilde{6^{k+1}S_j}}(b) - m_{\widetilde{S_j}}(b) \right| |K(x, y) - K(x, c_{S_j})| d\mu(x) \\
&=: H_1 + H_2.
\end{aligned}$$

By (2.8) and Lemma 2.9, we have

$$\begin{aligned}
(3.9) \quad \left| m_{\widetilde{6^{k+1}S_j}}(b) - m_{\widetilde{S_j}}(b) \right| &\lesssim \tilde{K}_{\widetilde{S_j}, \widetilde{6^{k+1}S_j}}^{(6)} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \\
&\lesssim \tilde{K}_{S_j, 6^{k+1}S_j}^{(6)} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \lesssim k \|b\|_{\widetilde{\text{RBMO}}(\mu)},
\end{aligned}$$

which, together with (3.2), implies that, for any  $y \in S_j$ ,

$$H_2 \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{k=1}^{\infty} k \int_{6^{k+1}S_j \setminus 6^k S_j} |K(x, y) - K(x, c_{S_j})| d\mu(x) \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}.$$

For  $H_1$ , from (3.5) and Proposition 2.16, we deduce that

$$\begin{aligned} H_1 &\lesssim \sum_{k=1}^{\infty} \mu(2 \cdot 6^{k+1} S_j) \left\| b - m_{\widetilde{6^{k+1} S_j}}(b) \right\|_{\exp L(\mu), 6^{k+1} S_j} \\ &\quad \times \left\| [K(\cdot, y) - K(\cdot, c_{S_j})] \chi_{6^{k+1} S_j \setminus 6^k S_j} \right\|_{L \log L(\mu), 6^{k+1} S_j} \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{k=1}^{\infty} \mu(2 \cdot 6^{k+1} S_j) \left\| [K(\cdot, y) - K(\cdot, c_{S_j})] \chi_{6^{k+1} S_j \setminus 6^k S_j} \right\|_{L \log L(\mu), 6^{k+1} S_j}. \end{aligned}$$

Choose

$$l_k := \left[ \mu(2 \cdot 6^{k+1} S_j) \right]^{-1} \left[ k \int_{6^{k+1} S_j \setminus 6^k S_j} |K(x, y) - K(x, c_{S_j})| d\mu(x) + 2^{-k} \right].$$

By (3.1), (2.2) and Lemma 2.3, we conclude that, for any  $y \in S_j$ ,

$$\begin{aligned} &\frac{1}{\mu(2 \cdot 6^{k+1} S_j)} \int_{6^{k+1} S_j \setminus 6^k S_j} \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} \log \left( 2 + \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} \right) d\mu(x) \\ &\lesssim \frac{1}{\mu(2 \cdot 6^{k+1} S_j)} \int_{6^{k+1} S_j \setminus 6^k S_j} \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} \\ &\quad \times \log \left( 2 + \frac{1}{l_k \lambda(x, d(x, y))} + \frac{1}{l_k \lambda(x, d(x, c_{S_j}))} \right) d\mu(x) \\ &\lesssim \frac{1}{\mu(2 \cdot 6^{k+1} S_j)} \int_{6^{k+1} S_j \setminus 6^k S_j} \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} \log \left( 2 + \frac{1}{l_k \lambda(c_{S_j}, d(x, c_{S_j}))} \right) d\mu(x) \\ &\lesssim \frac{1}{\mu(2 \cdot 6^{k+1} S_j)} \log \left( 2 + \frac{2^k \mu(2 \cdot 6^{k+1} S_j)}{\lambda(c_{S_j}, 6^k r_{S_j})} \right) \int_{6^{k+1} S_j \setminus 6^k S_j} \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} d\mu(x) \\ &\lesssim \frac{k}{\mu(2 \cdot 6^{k+1} S_j)} \int_{6^{k+1} S_j \setminus 6^k S_j} \frac{|K(x, y) - K(x, c_{S_j})|}{l_k} d\mu(x) \lesssim 1, \end{aligned}$$

which implies that

$$\left\| \{K(\cdot, y) - K(\cdot, c_{S_j})\} \chi_{6^{k+1} S_j \setminus 6^k S_j} \right\|_{L \log L(\mu), 6^{k+1} S_j} \lesssim l_k.$$

From this and (3.2), it follows that

$$\begin{aligned} H_1 &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{k=1}^{\infty} \mu(2 \cdot 6^{k+1} S_j) l_k \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{k=1}^{\infty} \left[ k \int_{6^{k+1} S_j \setminus 6^k S_j} |K(x, y) - K(x, c_{S_j})| d\mu(x) + 2^{-k} \right] \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}. \end{aligned}$$

Combining the estimates for  $H_1$  and  $H_2$ , we then obtain

$$H \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{j=1}^{\infty} \|h_j\|_{L^1(\mu)} \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{\tilde{H}^1(\mu)}.$$

We finally conclude that

$$\mu(\{x \in \mathcal{X} : |T_b^I f(x)| > t\}) \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} t^{-1} \|f\|_{\tilde{H}^1(\mu)},$$

which, together with the estimate (3.8), completes the proof of Theorem 3.1.  $\square$

**Remark 3.4.** Let  $b \in \text{RBMO}(\mu)$ . Fu et al. [9, Theorem 3.10] obtained the boundedness on Lebesgue spaces  $L^p(\mu)$  with  $p \in (1, \infty)$  of the commutator  $T_b$  generated by  $b$  and  $T$  with kernel satisfying (3.1) and the following *stronger regularity condition*, that is, there exist positive constants  $C$ ,  $\delta \in (0, 1]$  and  $c_{(K)}$ , depending on  $K$ , such that, for all  $x, \tilde{x}, y \in \mathcal{X}$  with  $d(x, y) \geq c_{(K)} d(x, \tilde{x})$ ,

$$(3.10) \quad |K(x, y) - K(\tilde{x}, y)| + |K(y, x) - K(y, \tilde{x})| \leq C \frac{[d(x, \tilde{x})]^\delta}{[d(x, y)]^\delta \lambda(x, d(x, y))}.$$

A new example of the operator with kernel satisfying (3.1) and (3.10) is the so-called Bergman-type operator appearing in [43]; see also [22] for an explanation. Notice that  $\text{RBMO}(\mu) \subset \widetilde{\text{RBMO}}(\mu)$ . Theorem 3.1 also holds true for the commutator  $T_b$  generated by  $b \in \widetilde{\text{RBMO}}(\mu)$  and  $T$  with kernel satisfying (3.1) and (3.10). Moreover, when  $b \in \widetilde{\text{RBMO}}(\mu)$  and  $T$  with kernel satisfying (3.1) and (3.2), we also prove that the commutator  $T_b$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ , which improves [9, Theorem 3.10]; see Section 5 below.

## 4 Boundedness of Commutators $T_{\alpha, b}$ on $\tilde{H}^1(\mu)$

In this section, we establish the boundedness from  $\tilde{H}^1(\mu)$  into  $L^{1/(1-\alpha), \infty}(\mu)$  of the commutator generated by the generalized fractional integral  $T_\alpha$  ( $\alpha \in (0, 1)$ ) and the  $\widetilde{\text{RBMO}}(\mu)$  function. We begin with the definition of the generalized fractional integral.

**Definition 4.1.** Let  $\alpha \in (0, 1)$ . A function  $K_\alpha \in L_{\text{loc}}^1(\{\mathcal{X} \times \mathcal{X}\} \setminus \{(x, x) : x \in \mathcal{X}\})$  is called a *generalized fractional integral kernel* if there exists a positive constant  $C_{(K_\alpha)}$ , depending only on  $K_\alpha$ , such that

(i) for all  $x, y \in \mathcal{X}$  with  $x \neq y$ ,

$$(4.1) \quad |K_\alpha(x, y)| \leq C_{(K_\alpha)} \frac{1}{[\lambda(x, d(x, y))]^{1-\alpha}};$$

(ii) there exist positive constant  $\delta \in (0, 1]$  and  $c_{(K_\alpha)} \in (0, \infty)$ , depending only on  $K_\alpha$ , such that, for all  $x, \tilde{x}, y \in \mathcal{X}$  with  $d(x, y) \geq c_{(K_\alpha)} d(x, \tilde{x})$ ,

$$(4.2) \quad |K_\alpha(x, y) - K_\alpha(\tilde{x}, y)| + |K_\alpha(y, x) - K_\alpha(y, \tilde{x})| \leq C_{(K_\alpha)} \frac{[d(x, \tilde{x})]^\delta}{[d(x, y)]^\delta [\lambda(x, d(x, y))]^{1-\alpha}}.$$

A linear operator  $T_\alpha$  is called a *generalized fractional integral* with kernel  $K_\alpha$  satisfying (4.1) and (4.2) if, for all  $f \in L_b^\infty(\mu)$  and  $x \notin \text{supp } f$ ,

$$(4.3) \quad T_\alpha f(x) := \int_{\mathcal{X}} K_\alpha(x, y) f(y) d\mu(y).$$

Let  $b \in \widetilde{\text{RBMO}}(\mu)$  and  $T_\alpha$  be the generalized fractional integral. The commutator  $T_{\alpha, b}$ , generated by  $b$  and  $T_\alpha$ , is defined by setting, for any suitable function  $f$ ,

$$(4.4) \quad T_{\alpha, b} f := b T_\alpha f - T_\alpha(bf).$$

Now we state the main result of this section as follows.

**Theorem 4.2.** *Let  $\alpha \in (0, 1)$  and  $b \in \widetilde{\text{RBMO}}(\mu)$ . Assume that the generalized fractional integral  $T_\alpha$ , defined by (4.3) associated with kernel  $K_\alpha$  satisfying (4.1) and (4.2), is bounded from  $L^p(\mu)$  into  $L^q(\mu)$  for all  $p \in (1, 1/\alpha)$  and  $1/q = 1/p - \alpha$ . Then the commutator  $T_{\alpha, b}$  defined by (4.4) is bounded from  $\widetilde{H}^1(\mu)$  into  $L^{1/(1-\alpha), \infty}(\mu)$ , that is, there exists a positive constant  $C$  such that, for all  $t \in (0, \infty)$  and all functions  $f \in \widetilde{H}^1(\mu)$ ,*

$$[\mu(\{x \in \mathcal{X} : |T_{\alpha, b} f(x)| > t\})]^{1-\alpha} \leq C \|b\|_{\widetilde{\text{RBMO}}(\mu)} t^{-1} \|f\|_{\widetilde{H}^1(\mu)}.$$

To prove Theorem 4.2, we need the following result from [10, Theorem 1.13].

**Lemma 4.3.** *Let  $\alpha \in (0, 1)$  and  $T_\alpha$  be as in (4.3) with kernel  $K_\alpha$  satisfying (4.1) and (4.2). Then the following statements are equivalent:*

- (i)  $T_\alpha$  is bounded from  $L^p(\mu)$  into  $L^q(\mu)$  for all  $p \in (1, 1/\alpha)$  and  $1/q = 1/p - \alpha$ ;
- (ii)  $T_\alpha$  is bounded from  $L^1(\mu)$  into  $L^{1/(1-\alpha), \infty}(\mu)$ .

*Proof of Theorem 4.2.* For each fixed  $f \in \widetilde{H}^1(\mu)$ , by Definition 2.11, we have a decomposition  $f = \sum_{j=1}^\infty h_j$ , where, for any  $j \in \mathbb{N}$ ,  $h_j$  is an  $(\infty, 2 - \alpha, 12)_\lambda$ -atomic block,  $\text{supp } h_j \subset S_j$ , and

$$\sum_{j=1}^\infty |h_j|_{\widetilde{H}_{\text{atb}, 12}^{1, \infty, 2-\alpha}(\mu)} \leq 2 \|f\|_{\widetilde{H}^1(\mu)}.$$

Moreover, for each fixed  $j$ , we can further decompose  $h_j$  as  $h_j = r_{j,1} a_{j,1} + r_{j,2} a_{j,2}$ , where, for any  $i \in \{1, 2\}$ ,  $r_{j,i} \in \mathbb{C}$ ,  $a_{j,i}$  is a bounded function supported on some ball  $B_{j,i} \subset S_j$  satisfying

$$\|a_{j,i}\|_{L^\infty(\mu)} \leq \left[ \mu(12B_{j,i}) \left\{ \widetilde{K}_{B_{j,i}, S_j}^{(12)} \right\}^{2-\alpha} \right]^{-1}$$

and  $|h_j|_{\widetilde{H}_{\text{atb}, 12}^{1, \infty, 2-\alpha}(\mu)} = |r_{j,1}| + |r_{j,2}|$ . By Lemma 2.10, we further conclude that, for any  $j$  and any  $i \in \{1, 2\}$ ,

$$(4.5) \quad \|a_{j,i}\|_{L^\infty(\mu)} \lesssim \left[ \mu(12B_{j,i}) \left\{ \widetilde{K}_{B_{j,i}, S_j}^{(6)} \right\}^{2-\alpha} \right]^{-1}.$$

Write

$$T_{\alpha,b}f = \sum_{j=1}^{\infty} [b - m_{\widetilde{S}_j}(b)] T_{\alpha}h_j + T_{\alpha} \left( \sum_{j=1}^{\infty} [m_{\widetilde{S}_j}(b) - b(\cdot)] h_j \right) =: T_{\alpha,b}^{\text{I}}f + T_{\alpha,b}^{\text{II}}f.$$

By Lemma 4.3 and an argument completely analogous to that used in the estimate for (3.8), we conclude that

$$(4.6) \quad [\mu(\{x \in \mathcal{X} : |T_{\alpha,b}^{\text{II}}f(x)| > t\})]^{1-\alpha} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} t^{-1} \|f\|_{\widetilde{H}^1(\mu)}.$$

We now estimate  $T_{\alpha,b}^{\text{I}}f(x)$ . By the Minkowski integral inequality and the Fatou lemma, we see that

$$\begin{aligned} & [\mu(\{x \in \mathcal{X} : |T_{\alpha,b}^{\text{I}}f(x)| > t\})]^{1-\alpha} \\ & \leq \left\{ \int_{\{x \in \mathcal{X} : |T_{\alpha,b}^{\text{I}}f(x)| > t\}} \frac{|\sum_{j=1}^{\infty} [b(x) - m_{\widetilde{S}_j}(b)] T_{\alpha}h_j(x)|^{\frac{1}{1-\alpha}}}{t^{\frac{1}{1-\alpha}}} d\mu(x) \right\}^{1-\alpha} \\ & \lesssim t^{-1} \sum_{j=1}^{\infty} \left[ \int_{\{x \in \mathcal{X} : |T_{\alpha,b}^{\text{I}}f(x)| > t\}} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}h_j(x)|^{\frac{1}{1-\alpha}} d\mu(x) \right]^{1-\alpha} \\ & \lesssim t^{-1} \sum_{j=1}^{\infty} \left[ \int_{6S_j} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}h_j(x)|^{\frac{1}{1-\alpha}} d\mu(x) \right]^{1-\alpha} + t^{-1} \sum_{j=1}^{\infty} \left[ \int_{\mathcal{X} \setminus 6S_j} \dots \right]^{1-\alpha} \\ & =: \text{G} + \text{H}. \end{aligned}$$

To estimate G, for each fixed  $j$ , write

$$\begin{aligned} & \int_{6S_j} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}h_j(x)|^{\frac{1}{1-\alpha}} d\mu(x) \\ & \lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \int_{6S_j} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}a_{j,1}(x)|^{\frac{1}{1-\alpha}} d\mu(x) \\ & \quad + |r_{j,2}|^{\frac{1}{1-\alpha}} \int_{6S_j} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}a_{j,2}(x)|^{\frac{1}{1-\alpha}} d\mu(x) =: \text{L}_{j,1} + \text{L}_{j,2}. \end{aligned}$$

We only consider the term  $\text{L}_{j,1}$ , the other term  $\text{L}_{j,2}$  can be estimated in a similar way, the details being omitted. Write

$$\begin{aligned} \text{L}_{j,1} & \lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \int_{6S_j \setminus 6B_{j,1}} \left| b(x) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}a_{j,1}(x)|^{\frac{1}{1-\alpha}} d\mu(x) \\ & \quad + |r_{j,1}|^{\frac{1}{1-\alpha}} \int_{6B_{j,1}} \left| b(x) - m_{\widetilde{6B_{j,1}}}(b) \right|^{\frac{1}{1-\alpha}} |T_{\alpha}a_{j,1}(x)|^{\frac{1}{1-\alpha}} d\mu(x) \\ & \quad + |r_{j,1}|^{\frac{1}{1-\alpha}} \left| m_{\widetilde{6B_{j,1}}}(b) - m_{\widetilde{S}_j}(b) \right|^{\frac{1}{1-\alpha}} \int_{6B_{j,1}} |T_{\alpha}a_{j,1}(x)|^{\frac{1}{1-\alpha}} d\mu(x) =: \text{U}_j + \text{V}_j + \text{W}_j. \end{aligned}$$

Let  $p \in (1, 1/\alpha)$  and  $1/q = 1/p - \alpha$ . Let  $\beta := q(1 - \alpha)$ . Then  $\beta \in (1, \infty)$ . Recall that  $\frac{1}{\beta} + \frac{1}{\beta'} = 1$ . It then follows, from the Hölder inequality, the assumption that  $T_\alpha$  is bounded from  $L^p(\mu)$  into  $L^q(\mu)$ , (2.8), Lemma 2.9 and (4.5), that

$$\begin{aligned} W_j &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{6B_{j,1}, \widetilde{S}_j}^{(6)} \right]^{\frac{1}{1-\alpha}} \left[ \int_{6B_{j,1}} |T_\alpha a_{j,1}(x)|^{\frac{1}{1-\alpha}\beta} d\mu(x) \right]^{\frac{1}{\beta}} [\mu(6B_{j,1})]^{\frac{1}{\beta'}} \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right]^{\frac{1}{1-\alpha}} \|T_\alpha a_{j,1}\|_{L^q(\mu)}^{\frac{q}{\beta}} [\mu(6B_{j,1})]^{\frac{1}{\beta'}} \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right]^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^p(\mu)}^{\frac{1}{1-\alpha}} [\mu(6B_{j,1})]^{\frac{1}{\beta'}} \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right]^{\frac{1}{1-\alpha}} [\mu(6B_{j,1})]^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}}, \end{aligned}$$

where, in the penultimate inequality, we used the fact that

$$(4.7) \quad \frac{1}{p(1-\alpha)} + \frac{1}{\beta'} = \frac{1}{1-\alpha}.$$

On the other hand, the Hölder inequality, together with Corollary 2.17, the boundedness from  $L^p(\mu)$  into  $L^q(\mu)$  of  $T_\alpha$ , (4.7) and (4.5), implies that

$$\begin{aligned} V_j &\lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \left[ \int_{6B_{j,1}} |b(x) - m_{\widetilde{6B_{j,1}}}(b)|^{\frac{1}{1-\alpha}\beta'} d\mu(x) \right]^{\frac{1}{\beta'}} \|T_\alpha a_{j,1}\|_{L^q(\mu)}^{\frac{q}{\beta}} \\ &\lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} [\mu(12B_{j,1})]^{\frac{1}{\beta'}} \|T_\alpha a_{j,1}\|_{L^q(\mu)}^{\frac{1}{1-\alpha}} \\ &\lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^p(\mu)}^{\frac{1}{1-\alpha}} [\mu(12B_{j,1})]^{\frac{1}{\beta'}} \\ &\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} [\mu(12B_{j,1})]^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}}. \end{aligned}$$

To estimate  $U_j$ , we first observe that, for any  $x \notin 6B_{j,1}$  and  $y \in B_{j,1}$ ,  $d(x, y) \sim d(x, c_{B_{j,1}})$ . It then follows from (4.1) that, for any  $x \notin 6B_{j,1}$ ,

$$\begin{aligned} |T_\alpha a_{j,1}(x)| &= \left| \int_{\mathcal{X}} K_\alpha(x, y) a_{j,1}(y) d\mu(y) \right| \leq \int_{B_{j,1}} |K_\alpha(x, y)| |a_{j,1}(y)| d\mu(y) \\ &\lesssim \frac{\|a_j\|_{L^1(\mu)}}{[\lambda(x, d(x, c_{B_{j,1}}))]^{1-\alpha}} \lesssim \frac{\|a_j\|_{L^\infty(\mu)} \mu(B_{j,1})}{[\lambda(x, d(x, c_{B_{j,1}}))]^{1-\alpha}}. \end{aligned}$$

Let  $N_1 := N_{6B_{j,1}, 6S_j}^{(6)} + [\log_6 2] + 1$ . A straightforward computation, via the above estimate, Corollary 2.17, (2.8), Lemma 2.9, (2.2) and (4.5), shows that

$$U_j \lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} [\mu(B_{j,1})]^{\frac{1}{1-\alpha}} \int_{6S_j \setminus 6B_{j,1}} \frac{|b(x) - m_{\widetilde{S}_j}(b)|^{\frac{1}{1-\alpha}}}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x)$$

$$\begin{aligned}
&\lesssim |r_{j,1}|^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} [\mu(B_{j,1})]^{\frac{1}{1-\alpha}} \sum_{k=1}^{N_1} \left[ \int_{6^{k+1}B_{j,1} \setminus 6^k B_{j,1}} \frac{|b(x) - m_{\widetilde{6^{k+1}B_{j,1}}}(b)|^{\frac{1}{1-\alpha}}}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x) \right. \\
&\quad \left. + \left| m_{\widetilde{6^{k+1}B_{j,1}}}(b) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} \int_{6^{k+1}B_{j,1} \setminus 6^k B_{j,1}} \frac{1}{\lambda(x, d(x, c_{B_{j,1}}))} d\mu(x) \right] \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} [\mu(B_{j,1})]^{\frac{1}{1-\alpha}} \\
&\quad \times \sum_{k=1}^{N_1} \left[ \frac{\mu(6^{k+2}B_{j,1})}{\lambda(c_{B_{j,1}}, 6^{k+2}r_{B_{j,1}})} + \left\{ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right\}^{\frac{1}{1-\alpha}} \frac{\mu(6^{k+1}B_{j,1})}{\lambda(c_{B_{j,1}}, 6^{k+1}r_{B_{j,1}})} \right] \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} [\mu(B_{j,1})]^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right]^{\frac{1}{1-\alpha}} \\
&\quad \times \left[ 1 + \sum_{k=1}^{N_1} \frac{\mu(6^k B_{j,1})}{\lambda(c_{B_{j,1}}, 6^k r_{B_{j,1}})} \right] \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}} \|a_{j,1}\|_{L^\infty(\mu)}^{\frac{1}{1-\alpha}} [\mu(B_{j,1})]^{\frac{1}{1-\alpha}} \left[ \widetilde{K}_{B_{j,1}, S_j}^{(6)} \right]^{\frac{2-\alpha}{1-\alpha}} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}}.
\end{aligned}$$

Combining the estimates for  $U_j$ ,  $V_j$  and  $W_j$ , we obtain

$$L_{j,1} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} |r_{j,1}|^{\frac{1}{1-\alpha}},$$

which further implies that

$$G \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{j=1}^{\infty} |h_j|_{\widetilde{H}_{\text{atb},12}^{1,\infty,2-\alpha}(\mu)} \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{\widetilde{H}^1(\mu)}.$$

It remains to estimate  $H$ . Observe that, for any  $x \notin 6S_j$  and  $y \in S_j$ ,  $d(x, y) \sim d(x, c_{S_j})$ . From this, together with the vanishing moment of  $h_j$  and (4.2), we deduce that, for any  $x \notin 6S_j$ ,

$$\begin{aligned}
|T_\alpha h_j(x)| &= \left| \int_{\mathcal{X}} K_\alpha(x, y) h_j(y) d\mu(y) \right| \leq \int_{S_j} |K_\alpha(x, y) - K_\alpha(x, c_{S_j})| |h_j(y)| d\mu(y) \\
&\lesssim \int_{S_j} \frac{[d(y, c_{S_j})]^\delta}{[d(x, y)]^\delta [\lambda(x, d(x, y))]^{1-\alpha}} |h_j(y)| d\mu(y) \\
&\lesssim \frac{(r_{S_j})^\delta}{[d(x, c_{S_j})]^\delta [\lambda(x, d(c_{S_j}, x))]^{1-\alpha}} \|h_j\|_{L^1(\mu)}.
\end{aligned}$$

On the other hand, a trivial computation, via Corollary 2.17, (3.9) and (2.1), gives us that, for any  $i \in \mathbb{N}$ ,

$$\frac{1}{\lambda(c_{S_j}, 6^i r_{S_j})} \int_{6^{i+1}S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} d\mu(x)$$



$$\begin{aligned}
&\leq \frac{1}{\lambda(c_{S_j}, 6^i r_{S_j})} \left[ \int_{6^{i+1} S_j} \left| b(x) - m_{\widetilde{6^{i+1} S_j}}(b) \right|^{\frac{1}{1-\alpha}} d\mu(x) \right. \\
&\quad \left. + \mu(6^{i+1} S_j) \left| m_{\widetilde{6^{i+1} S_j}}(b) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} \right] \\
&\lesssim \frac{\mu(2 \cdot 6^{i+1} S_j)}{\lambda(c_{S_j}, 6^i r_{S_j})} \|b\|_{\widetilde{\text{RBMO}}(\mu)}^{\frac{1}{1-\alpha}} + \frac{\mu(6^{i+1} S_j)}{\lambda(c_{S_j}, 6^i r_{S_j})} \left[ i \|b\|_{\widetilde{\text{RBMO}}(\mu)} \right]^{\frac{1}{1-\alpha}} \lesssim \left[ i \|b\|_{\widetilde{\text{RBMO}}(\mu)} \right]^{\frac{1}{1-\alpha}}.
\end{aligned}$$

By the above two estimates, we conclude that

$$\begin{aligned}
\text{H} &= t^{-1} \sum_{j=1}^{\infty} \left[ \int_{\mathcal{X} \setminus 6S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} \left| \int_{S_j} K_{\alpha}(x, y) h_j(y) d\mu(y) \right|^{\frac{1}{1-\alpha}} d\mu(x) \right]^{1-\alpha} \\
&\lesssim t^{-1} \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{\infty} \int_{6^{i+1} S_j \setminus 6^i S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} \right. \\
&\quad \left. \times \left[ \frac{(r_{S_j})^{\delta}}{(d(x, c_{S_j}))^{\delta} (\lambda(x, d(c_{S_j}, x)))^{1-\alpha}} \|h_j\|_{L^1(\mu)} \right]^{\frac{1}{1-\alpha}} d\mu(x) \right\}^{1-\alpha} \\
&\lesssim t^{-1} \sum_{j=1}^{\infty} \|h_j\|_{L^1(\mu)} \left[ \sum_{i=1}^{\infty} (6^i)^{\frac{-\delta}{1-\alpha}} \frac{1}{\lambda(c_{S_j}, 6^i r_{S_j})} \int_{6^{i+1} S_j} \left| b(x) - m_{\widetilde{S_j}}(b) \right|^{\frac{1}{1-\alpha}} d\mu(x) \right]^{1-\alpha} \\
&\lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \sum_{j=1}^{\infty} \|h_j\|_{L^1(\mu)} \left[ \sum_{i=1}^{\infty} \left( 6^{\frac{-i\delta}{1-\alpha}} \right) (i)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{\tilde{H}^1(\mu)}.
\end{aligned}$$

We finally obtain

$$\mu(\{x \in \mathcal{X} : |T_b^1 f(x)| > t\})^{1-\alpha} \lesssim t^{-1} \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{\tilde{H}^1(\mu)},$$

which, together with the estimate (4.6), completes the proof of Theorem 4.2.  $\square$

## 5 Boundedness of Commutators $T_b$ on $L^p(\mu)$ with $p \in (1, \infty)$

In this section, we establish the boundedness on  $L^p(\mu)$ , for all  $p \in (1, \infty)$ , of the commutator  $T_b$  generated by  $b \in \widetilde{\text{RBMO}}(\mu)$  and the Calderón-Zygmund operator  $T$  with kernel satisfying (3.1) and (3.2), which improves [9, Theorem 3.10].

**Theorem 5.1.** *Let  $b \in \widetilde{\text{RBMO}}(\mu)$ . Assume that the Calderón-Zygmund operator  $T$ , defined by (3.3) associated with kernel  $K$  satisfying (3.1) and (3.2), is bounded on  $L^2(\mu)$ . Then the commutator  $T_b$  defined in (3.4) is bounded on  $L^p(\mu)$  with  $p \in (1, \infty)$ .*

To prove Theorem 5.1, we borrow some ideas from the proof of [15, Theorem 4.3]. We need several tools, including an interpolation theorem for sublinear operators, which is an

extension of [15, Theorem 3.1] and whose proof is different from that of [15, Theorem 3.1]. We start with the notion of maximal functions in [2, 19].

For any  $f \in L^1_{loc}(\mu)$ , the *sharp maximal function*  $M^\sharp f$  is defined by setting, for all  $x \in \mathcal{X}$ ,

$$M^\sharp f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_{\tilde{B}}(f)| d\mu(y) + \sup_{\substack{x \in B \subset S \\ B, S \text{ } (6, \beta_6)\text{-doubling}}} \frac{|m_B(f) - m_S(f)|}{\tilde{K}_{B, S}^{(6)}},$$

where the first supremum is taking over all balls  $B$  containing  $x$ . By an argument similar to that used in the proof of [15, Lemma 3.1], we have the following result.

**Lemma 5.2.** *For  $r \in (0, 1)$  and any  $|f|^r \in L^1_{loc}(\mu)$ , let  $M_r^\sharp f := [M^\sharp(|f|^r)]^{\frac{1}{r}}$ . Then there exists a positive constant  $C_{(r)}$ , depending on  $r$ , such that, for all  $|f|^r \in L^1_{loc}(\mu)$ ,  $M_r^\sharp f \leq C_{(r)} M^\sharp f$ .*

The *non-centered doubling maximal operator*  $N$  is defined by setting, for all  $f \in L^1_{loc}(\mu)$  and  $x \in \mathcal{X}$ ,

$$Nf(x) := \sup_{\substack{B \ni x \\ B \text{ } (6, \beta_6)\text{-doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

By the Lebesgue differential theorem, it is easy to see that, for any  $f \in L^1_{loc}(\mu)$  and  $\mu$ -almost every  $x \in \mathcal{X}$ ,

$$(5.1) \quad |f(x)| \leq Nf(x)$$

(see [19, Corollary 3.6]). The following lemma is just [27, Lemma 3.3].

**Lemma 5.3.** *For all  $f \in L^1_{loc}(\mu)$ , with*

$$\int_{\mathcal{X}} f(y) d\mu(y) = 0 \quad \text{when } \mu(\mathcal{X}) < \infty,$$

*if  $\min\{1, Nf\} \in L^{p_0}(\mu)$  for some  $p_0 \in (1, \infty)$ , then, for all  $p \in [p_0, \infty)$ , there exists a positive constant  $C_{(p)}$ , depending on  $p$  but independent of  $f$ , such that*

$$\|Nf\|_{L^{p, \infty}(\mu)} \leq C_{(p)} \|M^\sharp f\|_{L^{p, \infty}(\mu)}.$$

Moreover, for  $r \in (0, \infty)$  and  $\eta \in (1, \infty)$ , the *maximal operator*  $M_{r, (\eta)}$  is defined by setting, for all  $f \in L^r_{loc}(\mu)$  and  $x \in \mathcal{X}$ ,

$$M_{r, (\eta)} f(x) := \sup_{B \ni x} \left[ \frac{1}{\mu(\eta B)} \int_B |f(y)|^r d\mu(y) \right]^{\frac{1}{r}},$$

where the supremum is taking over all balls  $B$  containing  $x$ . With a proof similar to that of [21, Lemma 2.3], we obtain the following useful properties of  $M_{r, (\eta)}$ , the details being omitted.

**Lemma 5.4.** *The following statements hold true:*

- (i) *Let  $p \in (1, \infty)$ ,  $r \in (1, p)$  and  $\eta \in [5, \infty)$ . Then  $M_{r,(\eta)}$  is bounded on  $L^p(\mu)$ .*
- (ii) *Let  $r \in (0, 1)$  and  $\eta \in [5, \infty)$ . Then  $M_{r,(\eta)}$  is bounded on  $L^{1,\infty}(\mu)$ , that is, there exists a positive constant  $C$  such that, for all  $f \in L^{1,\infty}(\mu)$ ,*

$$(5.2) \quad \sup_{\sigma>0} \sigma \mu(\{x \in \mathcal{X} : M_{r,(\eta)}f(x) > \sigma\}) \leq C \sup_{\sigma>0} \sigma \mu(\{x \in \mathcal{X} : |f(x)| > \sigma\}).$$

Now we state our interpolation theorem for sublinear operators as follows.

**Theorem 5.5.** *Let  $p_0 \in (1, \infty)$ ,  $k \in \mathbb{N}$  and  $T_i$  be a sublinear operator for all  $i \in \{1, \dots, k\}$ . Suppose that*

- (i)  *$T_1$  is bounded from  $\tilde{H}^1(\mu)$  into  $L^{1,\infty}(\mu)$ , that is, there exists a positive constant  $C$  such that, for all  $f \in \tilde{H}^1(\mu)$  and  $t \in (0, \infty)$ ,*

$$\mu(\{x \in \mathcal{X} : |T_1 f(x)| > t\}) \leq \frac{C}{t} \|f\|_{\tilde{H}^1(\mu)};$$

- (ii) *when  $\mu(\mathcal{X}) < \infty$ , for any  $p \in (1, \infty)$ , there exists a positive constant  $C_{(p)}$ , depending on  $p$ , such that, for all  $f \in L_b^\infty(\mu)$ ,*

$$\frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |T_1 f(x)| d\mu(x) \leq C \|f\|_{L^p(\mu)};$$

- (iii) *there exists a positive constant  $D$  such that, for all  $f \in L_b^\infty(\mu)$ ,*

$$M^\sharp(T_1 f) \leq \sum_{i=2}^k |T_i f| + D \|f\|_{L^\infty(\mu)};$$

- (iv)  *$T_i$  is bounded on  $L^{p_0}(\mu)$  for all  $i \in \{2, \dots, k\}$ .*

*Then  $T_1$  is bounded on  $L^p(\mu)$  for all  $p \in (1, p_0)$ .*

*Proof.* Let  $r \in (0, 1)$ . We first claim that, for all  $p \in (1, p_0)$ , there exists a positive constant  $C$  such that, for all  $f \in L_b^\infty(\mu)$ ,

$$(5.3) \quad \sup_{t>0} t^p \mu(\{x \in \mathcal{X} : |M_r^\sharp(T_1 f)(x)| > t\}) \leq C \|f\|_{L^p(\mu)}^p.$$

To show (5.3), notice that, although it is unclear whether the operator  $M_r^\sharp(T_1)$  is quasi-linear or not, we still conclude that there exists a positive constant  $\tilde{C}_{(r)}$ , depending on  $r$ , such that, for all  $f_1, f_2 \in L_b^\infty(\mu)$ ,

$$(5.4) \quad M_r^\sharp(T_1(f_1 + f_2)) \leq \tilde{C}_{(r)} \left[ M_{r,(6)}(T_1 f_1) + M_r^\sharp(T_1 f_2) \right].$$

Indeed, by  $r \in (0, 1)$ , we see that, for any  $f_1, f_2 \in L_b^\infty(\mu)$ ,  $x \in \mathcal{X}$  and ball  $B \ni x$ ,

$$(5.5) \quad \frac{1}{\mu(6B)} \int_B |T_1(f_1 + f_2)(y)|^r - m_{\tilde{B}}(|T_1(f_1 + f_2)|^r) d\mu(y)$$

$$\begin{aligned} &\leq \frac{1}{\mu(6B)} \int_B \left| |T_1 f_2(y)|^r - m_{\tilde{B}}(|T_1 f_2|^r) \right| d\mu(y) + m_{\tilde{B}}(|T_1 f_1|^r) \\ &\quad + \frac{1}{\mu(6B)} \int_B |T_1 f_1(y)|^r d\mu(y) \lesssim \left[ M_r^\sharp(T_1 f_2)(x) \right]^r + \left[ M_{r,(5)}(T_1 f_1)(x) \right]^r \end{aligned}$$

and, for any  $(6, \beta_6)$ -doubling balls  $B \subset S$  with  $B \ni x$ ,

$$\begin{aligned} (5.6) \quad &|m_B(|T_1(f_1 + f_2)|^r) - m_S(|T_1(f_1 + f_2)|^r)| \\ &\leq |m_B(|T_1 f_2|^r) - m_S(|T_1 f_2|^r)| + m_B(|T_1 f_1|^r) + m_S(|T_1 f_1|^r) \\ &\lesssim \tilde{K}_{B,S}^{(6)} \left[ M_r^\sharp(T_1 f_2)(x) \right]^r + \left[ M_{r,(6)}(T_1 f_1)(x) \right]^r. \end{aligned}$$

Combining (5.5) and (5.6), we obtain (5.4).

For each fixed  $t \in (0, \infty)$  and each  $f \in L_b^\infty(\mu)$ , applying the Calderón-Zygmund decomposition to  $|f|^p$  at level  $t^p$  (with  $t^p > \gamma_0 \|f\|_{L^p(\mu)} / \mu(\mathcal{X})$  if  $\mu(\mathcal{X}) < \infty$ ) with the notation as in Lemma 2.18, we obtain  $f = g + h$ , where  $g := f \chi_{\mathcal{X} \setminus (\cup_j 6B_j)} + \sum_j \varphi_j$  and  $h := f - g = \sum_j [\omega_j f - \varphi_j]$ . Moreover, we have

$$(5.7) \quad \|g\|_{L^\infty(\mu)} \leq (\gamma + 1)t \quad \text{and} \quad \|g\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}.$$

From this, together with (5.4), Lemma 5.2 and Theorem 5.5(iii), we deduce that

$$\begin{aligned} &\mu \left( \left\{ x \in \mathcal{X} : M_r^\sharp(T_1 f)(x) > 3D\tilde{C}_{(r)}C_{(r)}(\gamma + 1)t \right\} \right) \\ &\leq \mu \left( \left\{ x \in \mathcal{X} : M_r^\sharp(T_1 g)(x) > 2DC_{(r)}(\gamma + 1)t \right\} \right) \\ &\quad + \mu \left( \left\{ x \in \mathcal{X} : M_{r,(6)}(T_1 h)(x) > DC_{(r)}(\gamma + 1)t \right\} \right) \\ &\leq \mu \left( \left\{ x \in \mathcal{X} : M^\sharp(T_1 g)(x) > 2D(\gamma + 1)t \right\} \right) \\ &\quad + \mu \left( \left\{ x \in \mathcal{X} : M_{r,(6)}(T_1 h)(x) > DC_{(r)}(\gamma + 1)t \right\} \right) \\ &\leq \sum_{i=2}^k \mu \left( \left\{ x \in \mathcal{X} : |T_i g(x)| > D(\gamma + 1)t/(k - 1) \right\} \right) \\ &\quad + \mu \left( \left\{ x \in \mathcal{X} : M_{r,(6)}(T_1 h)(x) > DC_{(r)}(\gamma + 1)t \right\} \right). \end{aligned}$$

For  $i \in \{2, \dots, k\}$ , the boundedness of  $T_i$  on  $L^{p_0}(\mu)$  and (5.7) imply that

$$(5.8) \quad \mu(\{x \in \mathcal{X} : |T_i g(x)| > D(\gamma + 1)t\}) \lesssim t^{-p_0} \|T_i g\|_{L^{p_0}(\mu)}^{p_0} \lesssim t^{-p} \|f\|_{L^p(\mu)}^p.$$

By Lemma 5.4(ii), Theorem 5.5(i) and Lemma 2.18(iii), we have

$$\begin{aligned} \mu \left( \left\{ x \in \mathcal{X} : M_{r,(6)}(T_1 h)(x) > DC_{(r)}(\gamma + 1)t \right\} \right) &\lesssim t^{-1} \sup_{\sigma > 0} \sigma \mu(\{x \in \mathcal{X} : |T_1 h(x)| > \sigma\}) \\ &\lesssim t^{-1} \|h\|_{\tilde{H}^1(\mu)} \lesssim t^{-p} \|f\|_{L^p(\mu)}^p, \end{aligned}$$

which, together with the estimate (5.8), implies (5.3).

We now conclude the proof of Theorem 5.5 by considering the following two cases.

Case (i)  $\mu(\mathcal{X}) = \infty$ . Let

$$L_{b,0}^\infty(\mu) := \left\{ f \in L_b^\infty(\mu) : \int_{\mathcal{X}} f(x) d\mu(x) = 0 \right\}.$$

Then, in this case,  $L_{b,0}^\infty(\mu)$  is dense in  $L^p(\mu)$  for all  $p \in (1, \infty)$ . Let  $N_r f := [N(|f|^r)]^{1/r}$  for all  $f \in L_{b,0}^\infty(\mu)$ . We now show that, for all  $f \in L_{b,0}^\infty(\mu)$ ,  $\min\{1, N_r(T_1 f)\} \in L^p(\mu)$  for all  $p \in (1, \infty)$ . Indeed, for all  $f \in L_{b,0}^\infty(\mu)$ , we see that  $f \in \tilde{H}^1(\mu)$ . Moreover, by the definitions of  $N_r$  and  $M_{r,(6)}$  with  $r \in (0, 1)$ , we know that, for  $\mu$ -almost every  $x \in \mathcal{X}$ ,  $N_r(T_1 f)(x) \lesssim M_{r,(6)}(T_1 f)(x)$ . It then follows, from Lemma 5.4(ii) and Theorem 5.5(i), that, for all  $r \in (0, 1)$ ,

$$\|N_r(T_1 f)\|_{L^{1,\infty}(\mu)} \lesssim \|M_{r,(6)}(T_1 f)\|_{L^{1,\infty}(\mu)} \lesssim \|T_1 f\|_{L^{1,\infty}(\mu)} \lesssim \|f\|_{\tilde{H}^1(\mu)},$$

which implies that, for all  $p \in (1, \infty)$ ,

$$\begin{aligned} & \int_{\mathcal{X}} [\min\{1, N_r(T_1 f)\}(x)]^p d\mu(x) \\ &= p \int_0^2 t^{p-1} \mu(\{x \in \mathcal{X} : \min\{1, N_r(T_1 f)\}(x) > t\}) dt + p \int_2^\infty \dots \\ &\lesssim p \int_0^2 t^{p-1} \mu(\{x \in \mathcal{X} : \min\{1, N_r(T_1 f)\}(x) > t\}) dt \\ &\lesssim \|N_r(T_1 f)\|_{L^{1,\infty}(\mu)} \int_0^2 t^{p-2} dt \lesssim \|N_r(T_1 f)\|_{L^{1,\infty}(\mu)} < \infty. \end{aligned}$$

Thus, for all  $f \in L_{b,0}^\infty(\mu)$ ,  $\min\{1, N_r(T_1 f)\} \in L^p(\mu)$ . From this, (5.1), Lemma 5.3 and (5.3), we deduce that, for all  $f \in L_{b,0}^\infty(\mu)$  and all  $p \in (1, p_0)$ ,

$$\begin{aligned} \|T_1 f\|_{L^{p,\infty}(\mu)} &\leq \|N_r(T_1 f)\|_{L^{p,\infty}(\mu)} = \|N(|T_1 f|^r)\|_{L^{p/r,\infty}(\mu)}^{1/r} \lesssim \|M^\sharp(|T_1 f|^r)\|_{L^{p/r,\infty}(\mu)}^{1/r} \\ &\sim \|M_r^\sharp(T_1 f)\|_{L^{p,\infty}(\mu)} \lesssim \|f\|_{L^p(\mu)}, \end{aligned}$$

which, along with the Marcinkiewicz interpolation theorem and a standard density argument, implies that, for all  $p \in (1, p_0)$  and  $f \in L^p(\mu)$ ,

$$\|T_1 f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}.$$

Case (ii)  $\mu(\mathcal{X}) < \infty$ . In this case, for all  $r \in (0, 1)$ ,  $f \in L_b^\infty(\mu)$  and  $x \in \mathcal{X}$ , we see that

$$\begin{aligned} |T_1 f(x)| &\leq [N(|T_1 f|^r)(x)]^{1/r} \\ &\lesssim \left\{ N \left( |T_1 f|^r - \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |T_1 f(y)|^r d\mu(y) \right) (x) \right\}^{1/r} \\ &\quad + \left\{ \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |T_1 f(x)|^r d\mu(x) \right\}^{1/r} =: E(x) + F^{1/r}. \end{aligned}$$

Observe that  $\int_{\mathcal{X}} [|T_1 f(y)|^r - F] d\mu(y) = 0$  and, for all  $p \in (1, \infty)$ ,

$$\int_{\mathcal{X}} [\min\{1, N(|T_1 f|^r - F)(x)\}]^p d\mu(x) < \mu(\mathcal{X}) < \infty.$$

From this, together with Lemma 5.3,  $M^\sharp(F) = 0$  and (5.3), we deduce that, for all  $p \in (1, p_0)$ ,

$$\|N(|T_1 f|^r - F)\|_{L^{p/r, \infty}(\mu)}^{1/r} \lesssim \|M^\sharp(|T_1 f|^r)\|_{L^{p/r, \infty}(\mu)}^{1/r} \sim \|M_r^\sharp(T_1 f)\|_{L^{p, \infty}(\mu)} \lesssim \|f\|_{L^p(\mu)}.$$

A trivial computation via the Hölder inequality and Theorem 5.5(ii) leads to, for all  $p \in (1, \infty)$ ,

$$F^{1/r} \lesssim \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |T_1 f(x)| d\mu(x) \lesssim \|f\|_{L^p(\mu)}.$$

Combining the above two estimates, we obtain the desired conclusion also in this case, which completes the proof of Theorem 5.5.  $\square$

To prove Theorem 5.1, we also need the following pointwise estimate.

**Theorem 5.6.** *Let  $b \in L^\infty(\mu)$ , the operator  $T$  with kernel  $K$  be the same as in Theorem 3.1 and  $T_b$  as in (3.4). Suppose that  $T$  is bounded on  $L^2(\mu)$ . Then, for any  $s \in (1, \infty)$ , there exists a positive constant  $C_{(s)}$ , depending on  $s$ , such that, for all  $f \in L_b^\infty(\mu)$ ,*

$$(5.9) \quad M^\sharp(T_b f) \leq C_{(s)} \|b\|_{\widetilde{\text{RBMO}}(\mu)} [\|f\|_{L^\infty(\mu)} + M_{s,(5)} f + M_{s,(6)} T f + T_* f],$$

where  $T_*$  denotes the maximal Calderón-Zygmund operator defined by setting, for all  $f \in L_b^\infty(\mu)$  and  $x \in \mathcal{X}$ ,

$$T_* f(x) := \sup_{\varepsilon > 0} \left| \int_{d(x,y) > \varepsilon} K(x,y) f(y) d\mu(y) \right|.$$

To prove Theorem 5.6, we begin with the following technical lemma from [9, Lemma 3.13].

**Lemma 5.7.** *There exists a positive constant  $P_0$  (big enough), depending on  $C_{(\lambda)}$  in (2.1) and  $\beta_6$  as in (2.3), such that, if  $x \in \mathcal{X}$  is some fixed point and  $\{f_B\}_{B \ni x}$  is a collection of numbers such that  $|f_B - f_S| \leq C_{(x)}$  for all doubling balls  $B \subset S$  with  $x \in B$  such that  $\tilde{K}_{B,S}^{(6)} \leq P_0$ , then there exists a positive constant  $C$ , depending only on  $C_{(\lambda)}$ ,  $\beta_6$  and  $P_0$ , such that, for all doubling balls  $B \subset S$  with  $x \in B$ ,*

$$|f_B - f_S| \leq C \tilde{K}_{B,S}^{(6)} C_{(x)}.$$

*Proof of Theorem 5.6.* We first show that, for all  $x$  and balls  $B$  with  $B \ni x$ ,

$$(5.10) \quad \frac{1}{\mu(6B)} \int_B |T_b f(y) - h_B| d\mu(y)$$

$$\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} [M_{s,(5)}f(x) + M_{s,(6)}Tf(x) + \|f\|_{L^\infty(\mu)}]$$

and, for all balls  $B \subset S$  with  $B \ni x$ ,

$$(5.11) \quad |h_B - h_S| \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \left[ \tilde{K}_{B,S}^{(6)} \right]^2 [M_{s,(5)}f(x) + T^*f(x) + \|f\|_{L^\infty(\mu)}],$$

where

$$h_B := -m_B (T([b - m_{\tilde{B}}(b)]f\chi_{\mathcal{X} \setminus (6/5)B}))$$

and

$$h_S := -m_S (T([b - m_{\tilde{S}}(b)]f\chi_{\mathcal{X} \setminus (6/5)S})).$$

The hypotheses  $b \in L^\infty(\mu)$  and  $f \in L_b^\infty(\mu)$  imply that  $h_B$  and  $h_S$  are both finite.

The proof of (5.10) is analogous to that of [15, (10)] with a slight modification, the details being omitted.

We now show (5.11). For any two balls  $B \subset S$  with  $B \ni x$ , let  $N_2 := N_{B,S}^{(6)} + \lfloor \log_\rho 2 \rfloor + 2$ . Write

$$\begin{aligned} |h_B - h_S| &= |m_B (T([b - m_{\tilde{B}}(b)]f\chi_{\mathcal{X} \setminus (6/5)B})) - m_S (T([b - m_{\tilde{S}}(b)]f\chi_{\mathcal{X} \setminus (6/5)S}))| \\ &\leq |m_B (T([b - m_{\tilde{B}}(b)]f\chi_{6B \setminus (6/5)B}))| + |m_B (T([m_{\tilde{B}}(b) - m_{\tilde{S}}(b)]f\chi_{\mathcal{X} \setminus 6B}))| \\ &\quad + |m_B (T([b - m_{\tilde{S}}(b)]f\chi_{6^{N_2}B \setminus 6B}))| \\ &\quad + |m_B (T([b - m_{\tilde{S}}(b)]f\chi_{\mathcal{X} \setminus 6^{N_2}B})) - m_S (T([b - m_{\tilde{S}}(b)]f\chi_{\mathcal{X} \setminus 6^{N_2}B}))| \\ &\quad + |m_S (T([b - m_{\tilde{S}}(b)]f\chi_{6^{N_2}B \setminus (6/5)S}))| \\ &=: M_1 + M_2 + M_3 + M_4 + M_5. \end{aligned}$$

By a slight modified argument similar to that used in the proof of [2, Theorem 7.6], we conclude that, for all  $x \in \mathcal{X}$ ,

$$M_1 + M_5 \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} M_{s,(5)}f(x)$$

and

$$M_3 \lesssim \left[ \tilde{K}_{B,S}^{(6)} \right]^2 \|b\|_{\widetilde{\text{RBMO}}(\mu)} M_{s,(5)}f(x).$$

To estimate  $M_2$ , for  $x, y \in B$ , write

$$|T(f\chi_{\mathcal{X} \setminus 6B})(y)| \leq |T(f\chi_{\mathcal{X} \setminus 6B})(y) - T(f\chi_{\mathcal{X} \setminus 6B})(x)| + |T(f\chi_{\mathcal{X} \setminus 6B})(x)| =: \text{I} + \text{II}.$$

From (3.2), we deduce that

$$\begin{aligned} \text{I} &\leq \int_{\mathcal{X} \setminus 6B} |K(y, z) - K(x, z)| |f(z)| d\mu(z) \\ &\leq \sum_{k=1}^{\infty} \int_{6^{k+1}B \setminus 6^k B} |K(y, z) - K(x, z)| |f(z)| d\mu(z) \lesssim \|f\|_{L^\infty(\mu)}. \end{aligned}$$



The definition of  $T_*$ , together with (3.1), the fact that  $d(x, z) \sim d(c_B, z)$  for  $x \in B$  and  $z \in \mathcal{X}$  with  $d(z, x) > 2r_B$  and (2.2), implies that

$$\begin{aligned}
\Pi &= \left| \int_{\{z \in \mathcal{X}: d(z, x) > 2r_B\}} K(x, z) f(z) d\mu(z) - \int_{\{z \in 6B: d(z, x) > 2r_B\}} K(x, z) f(z) d\mu(z) \right| \\
&\lesssim \left| \int_{\{z \in \mathcal{X}: d(z, x) > 2r_B\}} K(x, z) f(z) d\mu(z) \right| + \int_{\{z \in 6B: d(z, x) > 2r_B\}} \frac{1}{\lambda(x, d(x, z))} |f(z)| d\mu(z) \\
&\lesssim T_* f(x) + \int_{\{z \in 6B: d(z, x) > 2r_B\}} \frac{1}{\lambda(c_B, d(z, c_B))} |f(z)| d\mu(z) \\
&\lesssim T_* f(x) + \frac{1}{\lambda(c_B, r_B)} \int_{6B} |f(z)| d\mu(z) \lesssim T_* f(x) + M_{s, (5)} f(x).
\end{aligned}$$

Thus, for  $x, y \in B$ , we have

$$|T(f \chi_{\mathcal{X} \setminus 6B})(y)| \lesssim \|f\|_{L^\infty(\mu)} + T_* f(x) + M_{s, (5)} f(x),$$

which, together with (2.8) and Lemma 2.9, shows that

$$\begin{aligned}
M_2 &= |m_B(T([m_{\bar{B}}(b) - m_{\bar{S}}(b)] f \chi_{\mathcal{X} \setminus 6B}))| \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \tilde{K}_{B, S}^{(6)} [\|f\|_{L^\infty(\mu)} + T_* f(x) + M_{s, (5)} f(x)].
\end{aligned}$$

Finally, we deal with the term  $M_4$ . As in the treatment for the term H in the proof of Theorem 3.1, an argument involving the generalization of the Hölder inequality (see Lemma 3.3) gives us that, for any  $y, z \in S$ ,

$$\begin{aligned}
&\left| T\left([b - m_{\bar{S}}(b)] f \chi_{\mathcal{X} \setminus 6^{N_2} B}\right)(y) - T\left([b - m_{\bar{S}}(b)] f \chi_{\mathcal{X} \setminus 6^{N_2} B}\right)(z) \right| \\
&\leq \int_{\mathcal{X} \setminus 2S} |K(y, w) - K(z, w)| |b(w) - m_{\bar{S}}(b)| |f(w)| d\mu(w) \\
&\leq \|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} \int_{6^k 2S \setminus 6^{k-1} 2S} |K(y, w) - K(z, w)| |b(w) - m_{\bar{S}}(b)| d\mu(w) \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{L^\infty(\mu)} \\
&\quad \times \sum_{k=1}^{\infty} \left[ k \int_{6^k 2S \setminus 6^{k-1} 2S} |K(y, w) - K(z, w)| d\mu(w) + 2^{-k} \right] \\
&\lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{L^\infty(\mu)}.
\end{aligned}$$

Taking the mean over  $B$  and  $S$  for  $y$  and  $z$ , respectively, we obtain

$$M_4 \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{L^\infty(\mu)}.$$

Combining the estimates for  $M_1, M_2, M_3, M_4$  and  $M_5$ , we obtain the desired estimate (5.11).

By an argument similar to that used in the proof of [38, Theorem 9.1] (see also the proof of [2, Theorem 7.6]), together with Lemma 5.7, (5.10) and (5.11), we obtain (5.9), which completes the proof of Theorem 5.6.  $\square$

We finally give the proof of Theorem 5.1.

*Proof of Theorem 5.1.* We first show that, if the Calderón-Zygmund operator  $T$  with kernel satisfying (3.1) and (3.2) is bounded on  $L^2(\mu)$ , then  $T_*$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ . Indeed, Liu et al. [28] proved that, if  $T$  with kernel satisfying (3.1) and the Hörmander condition, that is, there exists a positive constant  $C$  such that, for all  $x, \tilde{x} \in \mathcal{X}$  with  $x \neq \tilde{x}$ ,

$$\int_{d(x,y) \geq 2d(x,\tilde{x})} [|K(x,y) - K(\tilde{x},y)| + |K(y,x) - K(y,\tilde{x})|] d\mu(y) \leq C,$$

is bounded on  $L^2(\mu)$ , then the corresponding maximal operator  $T_*$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ . Since the Hörmander-type condition (3.2) is slightly stronger than the above Hörmander condition, we obtain the desired result.

On the other hand, by Lemma 3.2(ii) and Lemma 5.4(i), we conclude that, for all  $p \in (1, \infty)$  and  $s \in (1, p)$ ,  $M_{s,(6)} \circ T$  is bounded on  $L^p(\mu)$ .

Now we assume that  $b$  is bounded and consider the following two cases for  $\mu(\mathcal{X})$ .

*Case (i)*  $\mu(\mathcal{X}) = \infty$ . In this case, from the fact that, for all  $p \in (1, \infty)$  and  $s \in (1, p)$ ,  $M_{s,(5)}$ ,  $M_{s,(6)} \circ T$  and  $T_*$  are bounded on  $L^p(\mu)$  and Theorems 3.1, 5.6 and 5.5, we deduce that  $T_b$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$ .

*Case (ii)*  $\mu(\mathcal{X}) < \infty$ . In this case, by Corollary 2.17 and the Lebesgue dominated convergence theorem, we find that, for all  $r \in (1, \infty)$ ,

$$(5.12) \quad \left[ \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |b(x) - m_{\mathcal{X}}(b)|^r d\mu(x) \right]^{1/r} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}.$$

Write

$$|T_b f| \leq |[b - m_{\mathcal{X}}(b)] T f| + |T([b - m_{\mathcal{X}}(b)] f)|.$$

Then, for all  $p \in (1, \infty)$ , from the Hölder inequality, (5.12) and the boundedness of  $T$  on  $L^q(\mu)$  for all  $q \in (1, p]$ , it follows that

$$\frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |T_b f(x)| d\mu(x) \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)} \|f\|_{L^p(\mu)},$$

which, together with Theorems 3.1, 5.6 and 5.5, implies that  $T_b$  is also bounded on  $L^p(\mu)$  for all  $p \in (1, \infty)$  in this case.

If  $b$  is not bounded, let  $q \in (0, \infty)$  and, for all  $x \in \mathcal{X}$ ,

$$b_q(x) := \begin{cases} b(x), & \text{if } |b(x)| \leq q, \\ q \frac{b(x)}{|b(x)|}, & \text{if } |b(x)| > q. \end{cases}$$

By an argument similar to that used in the proof of [9, Lemma 3.11], we see that  $b_q \in \widetilde{\text{RBMO}}(\mu)$  and  $\|b_q\|_{\widetilde{\text{RBMO}}(\mu)} \lesssim \|b\|_{\widetilde{\text{RBMO}}(\mu)}$ , which, together with a standard limit argument, completes the proof of Theorem 5.1.  $\square$

## References

- [1] T. A. Bui, Boundedness of maximal operators and maximal commutators on non-homogeneous spaces, in: CMA Proceedings of AMSI International Conference on Harmonic Analysis and Applications (Macquarie University, February 2011), Vol. 45, pp. 22-36, Macquarie University, Australia, 2013.
- [2] T. A. Bui and X. T. Duong, Hardy spaces, regularized BMO spaces and the boundedness of Calderón-Zygmund operators on non-homogeneous spaces, *J. Geom. Anal.* 23 (2013), 895-932.
- [3] Y. Cao and J. Zhou, Morrey spaces for nonhomogeneous metric measure spaces, *Abstr. Appl. Anal.* 2013, Art. ID 196459, 8 pp.
- [4] J. Chen, X. Chen and F. Jin, Endpoint estimates for generalized multilinear fractional integrals on the nonhomogeneous metric spaces, *Chin. Ann. Math. B* (to appear).
- [5] W. Chen, Y. Meng and D. Yang, Calderón-Zygmund operators on Hardy spaces without the doubling condition, *Proc. Amer. Math. Soc.* 133 (2005), 2671-2680.
- [6] W. Chen and E. Sawyer, A note on commutators of fractional integrals with  $RBMO(\mu)$  functions, *Illinois J. Math.* 46 (2002), 1287-1298.
- [7] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative Sur Certains Espaces Homogènes*. (French) *Étude de Certaines Intégrales Singulières*, Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.
- [8] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977), 569-645.
- [9] X. Fu, D. Yang and W. Yuan, Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous spaces, *Taiwanese J. Math.* 16 (2012), 2203-2238.
- [10] X. Fu, D. Yang and W. Yuan, Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces, *Taiwanese J. Math.* 18 (2014), 509-557.
- [11] X. Fu, Da. Yang and Do. Yang, The molecular characterization of the Hardy space  $H^1$  on non-homogeneous metric measure spaces and its application, *J. Math. Anal. Appl.* 410 (2014), 1028-1042.
- [12] X. Fu, H. Lin, Da. Yang and Do. Yang, Hardy spaces  $H^p$  over non-homogeneous metric measure spaces and their applications, *Sci. China Math.* 58 (2015), 309-388.
- [13] V. Guliyev and Y. Sawano, Linear and sublinear operators on generalized Morrey spaces with non-doubling measures, *Publ. Math. Debrecen* 83 (2013), 303-327.
- [14] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York, 2001.
- [15] G. Hu, Y. Meng and D. Yang, New atomic characterization of  $H^1$  space with non-doubling measures and its applications, *Math. Proc. Cambridge Philos. Soc.* 138 (2005), 151-171.
- [16] G. Hu, Y. Meng and D. Yang, Multilinear commutators of singular integrals with non doubling measures, *Integral Equations Operator Theory* 51 (2005), 235-255.
- [17] G. Hu, Y. Meng and D. Yang, A new characterization of regularized BMO spaces on non-homogeneous spaces and its applications, *Ann. Acad. Sci. Fenn. Math.* 38 (2013), 3-27.

- [18] G. Hu, Da. Yang and Do. Yang,  $h^1$ , bmo, blo and Littlewood-Paley  $g$ -functions with non-doubling measures, *Rev. Mat. Iberoam.* 25 (2009), 595-667.
- [19] T. Hytönen, A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa, *Publ. Mat.* 54 (2010), 485-504.
- [20] T. Hytönen, Da. Yang and Do. Yang, The Hardy space  $H^1$  on non-homogeneous metric spaces, *Math. Proc. Cambridge Philos. Soc.* 153 (2012), 9-31.
- [21] T. Hytönen, S. Liu, Da. Yang and Do. Yang, Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces, *Canad. J. Math.* 64 (2012), 892-923.
- [22] T. Hytönen and H. Martikainen, Non-homogeneous  $Tb$  theorem and random dyadic cubes on metric measure spaces, *J. Geom. Anal.* 22 (2012), 1071-1107.
- [23] T. Hytönen and H. Martikainen, Non-homogeneous  $T1$  theorem for bi-parameter singular integrals, *Adv. Math.* 261 (2014), 220-273.
- [24] H. Lin and D. Yang, Spaces of type BLO on non-homogeneous metric measures, *Front. Math. China* 6 (2011), 271-292.
- [25] H. Lin and D. Yang, An interpolation theorem for sublinear operators on non-homogeneous metric measure spaces, *Banach J. Math. Anal.* 6 (2012), 168-179.
- [26] H. Lin and D. Yang, Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces, *Sci. China Math.* 57 (2014), 123-144.
- [27] S. Liu, Da. Yang and Do. Yang, Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: equivalent characterizations, *J. Math. Anal. Appl.* 386 (2012), 258-272.
- [28] S. Liu, Y. Meng and D. Yang, Boundedness of maximal Calderón-Zygmund operators on non-homogeneous metric measure spaces, *Proc. Roy. Soc. Edinburgh Sect. A* 144 (2014), 567-589.
- [29] F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* 1998, 463-487.
- [30] F. Nazarov, S. Treil and A. Volberg, The  $Tb$ -theorem on non-homogeneous spaces, *Acta Math.* 190 (2003), 151-239.
- [31] C. Pérez and R. Trujillo-González, Sharp weighted estimates for multilinear commutators, *London Math. Soc. (2)* 65 (2002), 672-692.
- [32] Y. Sawano and T. Shimomura, Sobolev embeddings for Riesz potentials of functions in Musielak-Orlicz-Morrey spaces over non-doubling measure spaces, *Integral Transform. Spec. Funct.* 25 (2014), 976-991.
- [33] Y. Sawano, T. Shimomura and H. Tanaka, A remark on modified Morrey spaces on metric measure spaces, *Hokkaido Math. J.* (to appear).
- [34] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, *Acta Math. Sin. (Engl. Ser.)* 21 (2005), 1535-1544.
- [35] Y. Sawano and H. Tanaka, Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures, *Taiwanese J. Math.* 11 (2007), 1091-1112.
- [36] C. Tan and J. Li, Littlewood-Paley theory on metric measure spaces with non doubling measures and its applications, *Sci. China Math.* 58 (2015), 983-1004.

- [37] C. Tan and J. Li, Some remarks on upper doubling metric measure spaces, *Math. Nachr.* (to appear).
- [38] X. Tolsa, BMO,  $H^1$ , and Calderón-Zygmund operators for non doubling measures, *Math. Ann.* 319 (2001), 89-149.
- [39] X. Tolsa, Littlewood-Paley theory and the  $T(1)$  theorem with non-doubling measures, *Adv. Math.* 164 (2001), 57-116.
- [40] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, *Acta Math.* 190 (2003), 105-149.
- [41] X. Tolsa, The space  $H^1$  for nondoubling measures in terms of a grand maximal operator, *Trans. Amer. Math. Soc.* 355 (2003), 315-348.
- [42] X. Tolsa, Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory, *Progress in Mathematics*, 307, Birkhäuser/Springer, Cham, 2014. xiv+396 pp.
- [43] A. Volberg and B. D. Wick, Bergman-type singular operators and the characterization of Carleson measures for Besov-Sobolev spaces on the complex ball, *Amer. J. Math.* 134 (2012), 949-992.
- [44] R. Xie, H. Gong and X. Zhou, Commutators of multilinear singular integral operators on non-homogeneous metric measure spaces, *Taiwanese J. Math.* 19 (2015), 703-723.
- [45] Da. Yang, Do. Yang and X. Fu, The Hardy space  $H^1$  on non-homogeneous spaces and its applications—a survey, *Eurasian Math. J.* 4 (2013), 104-139.
- [46] Da. Yang, Do. Yang and G. Hu, The Hardy Space  $H^1$  with Non-doubling Measures and Their Applications, *Lecture Notes in Mathematics* 2084, Springer-Verlag, Berlin, 2013. xiii+653 pp.

Haibo Lin

College of Science, China Agricultural University, Beijing 100083, People's Republic of China

*E-mail:* haibolincau@126.com (H. Lin)

Suqing Wu and Dachun Yang (Corresponding author)

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

*E-mails:* wusq@mail.bnu.edu.cn (S. Wu)

dcyang@bnu.edu.cn (D. Yang)